USC Viterbi School of Engineering

Configuration Spaces and Kinematic Transformations

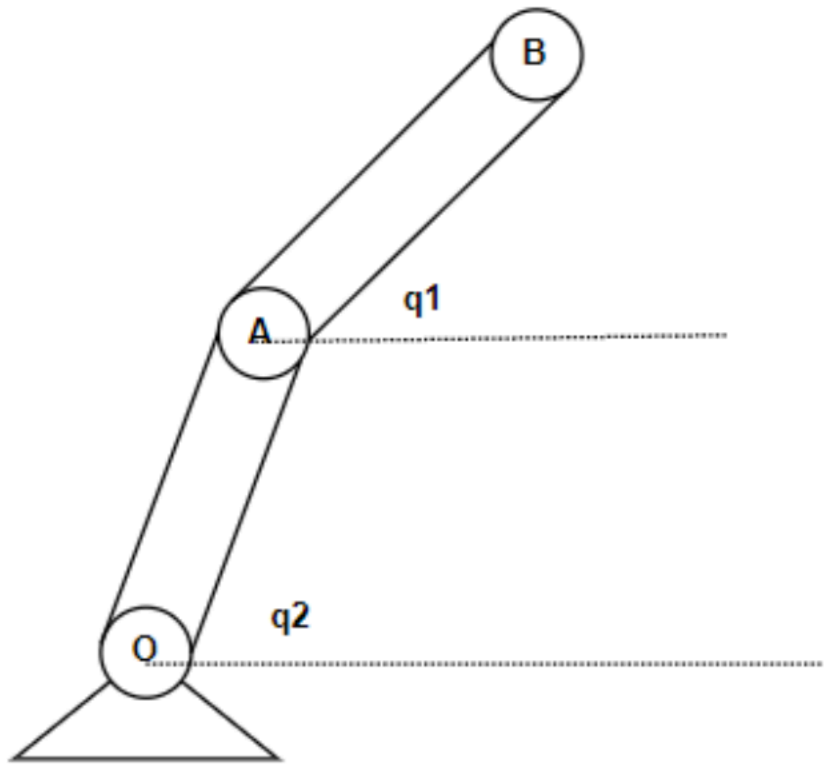
CSCI 545 Introduction to Robotics
Instructor: Stefanos Nikolaidis

Problem

- How can we represent a set of possible transformation that can be applied to the robot?
- We need a representation that is compact enough to be agnostic of the robot's geometry.

Example

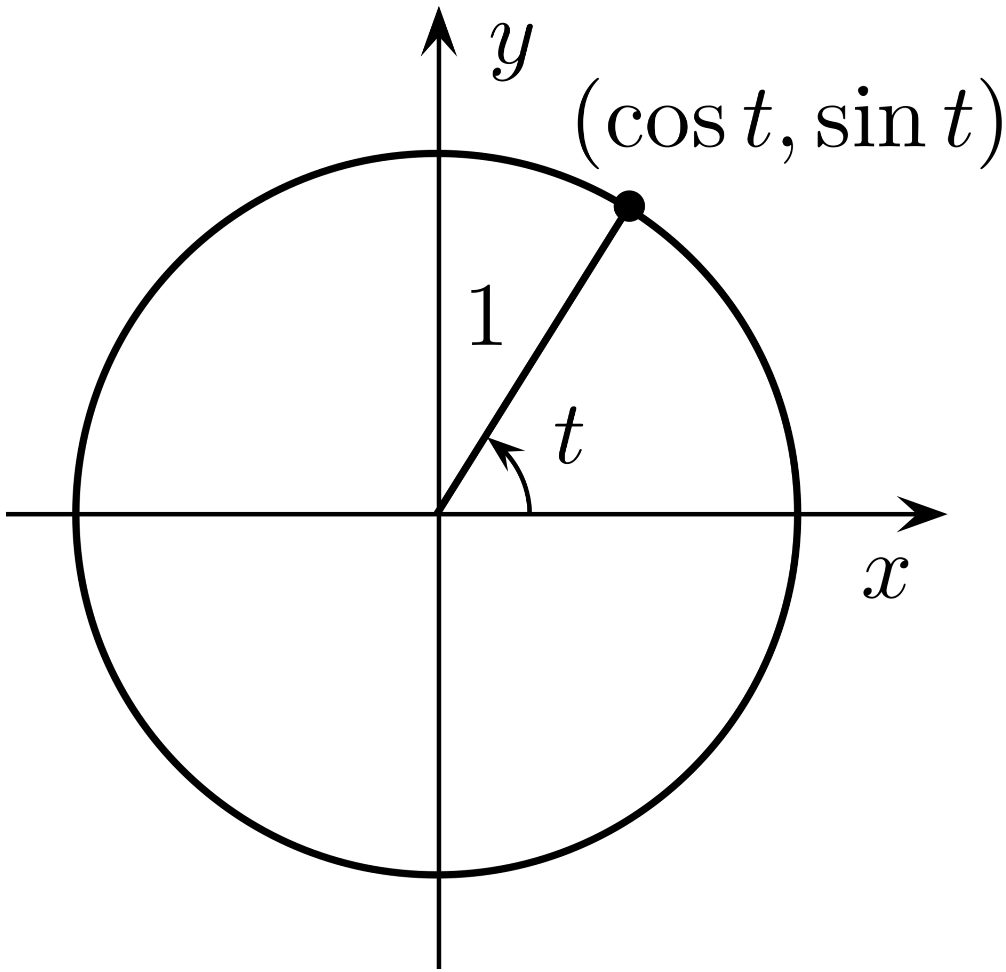
Example



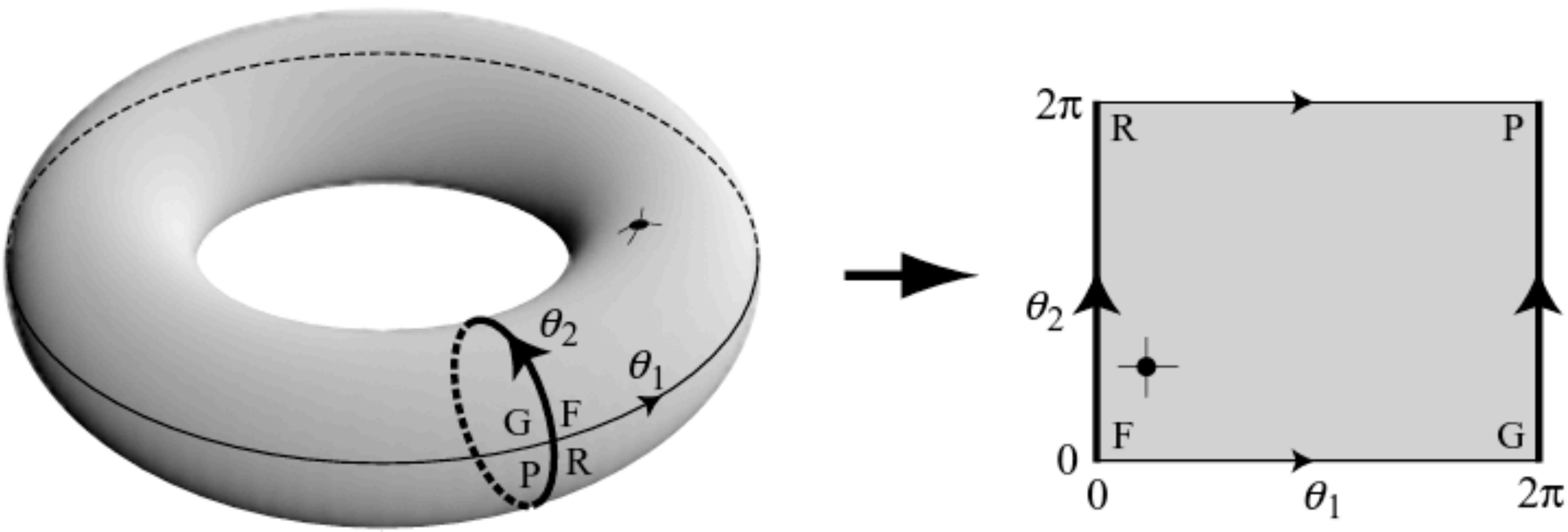
Definition

- Configuration space: A topological space , where every element x in Q corresponds to a valid configuration of the robot. Each configuration of the robot can be identified with a unique element of Q.

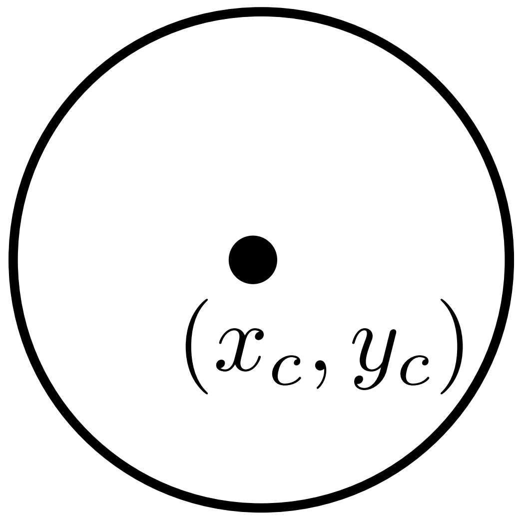
Example: Space S^1



Example: Space $S^1 \times S^1$



Example: Circular Mobile Robot



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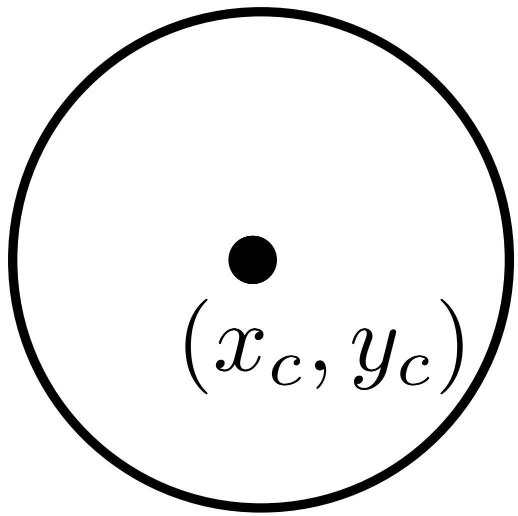

$$R(x, y) = \{(x, y) | (x - x_c)^2 + (y - y_c)^2 \leq r^2\}$$

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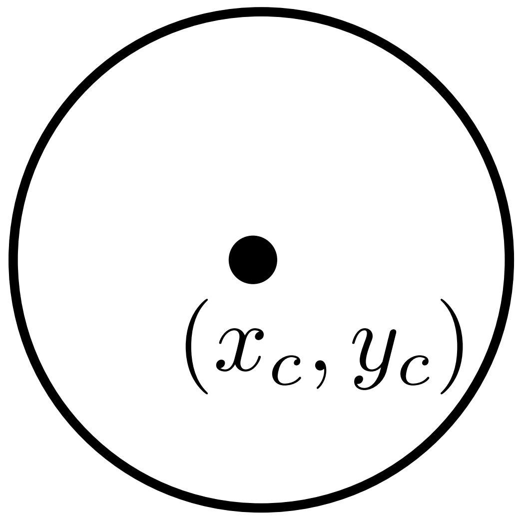

$$(x - x_c)^2 + (y - y_c)^2 \leq r^2$$

Path Planning Problem

Continuous mapping $\tau : [0, 1] \rightarrow Q$, such that no configuration in the path causes a collision between the robot and the obstacle.

Configuration Space Obstacle

$$QO_i : \{q \in Q | R(q) \bigcap WOi \neq \emptyset\}$$

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$$Q_{free} = Q \setminus (\cup_i QO_i)$$

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Path Planning Problem

$$\tau : [0, 1] \rightarrow Q_{free}$$

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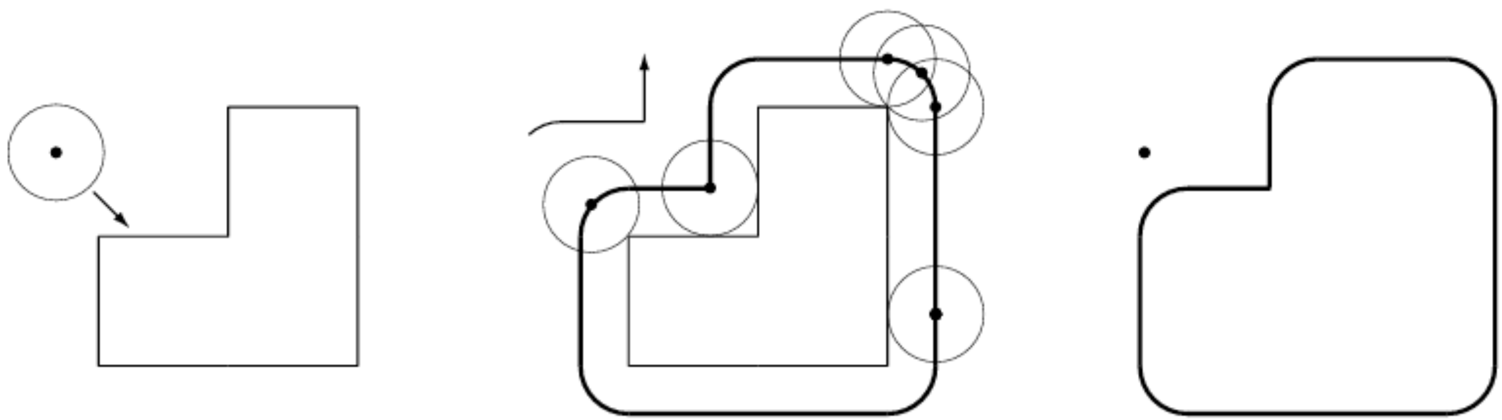

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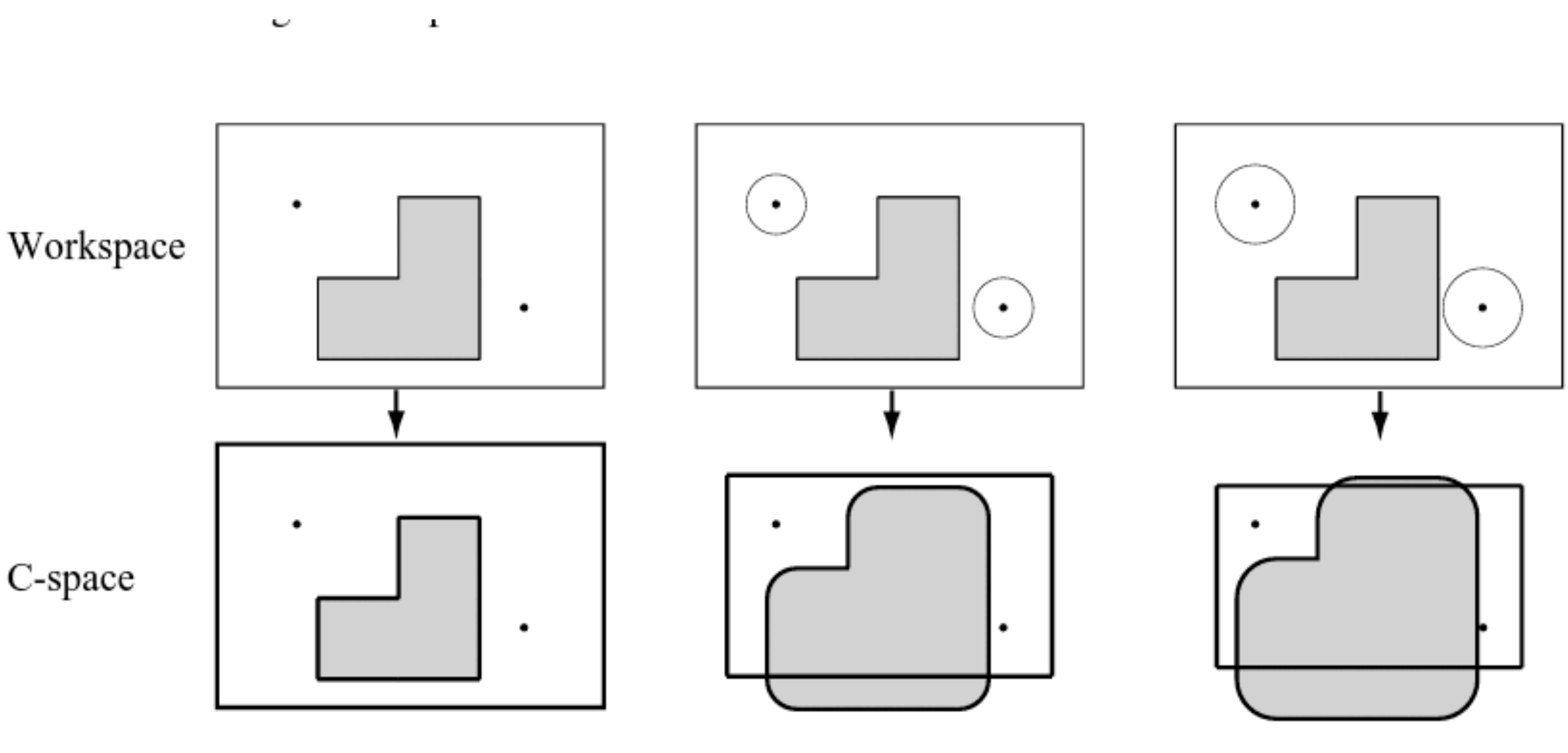
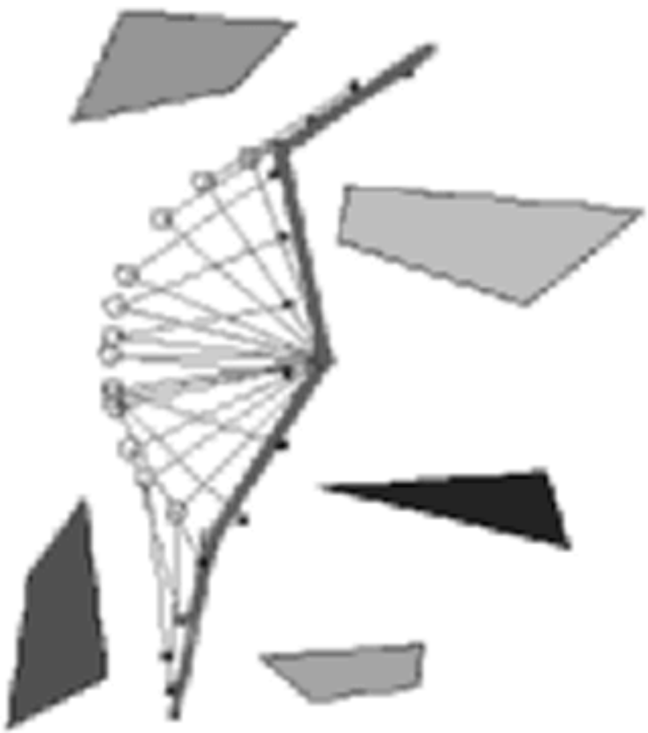
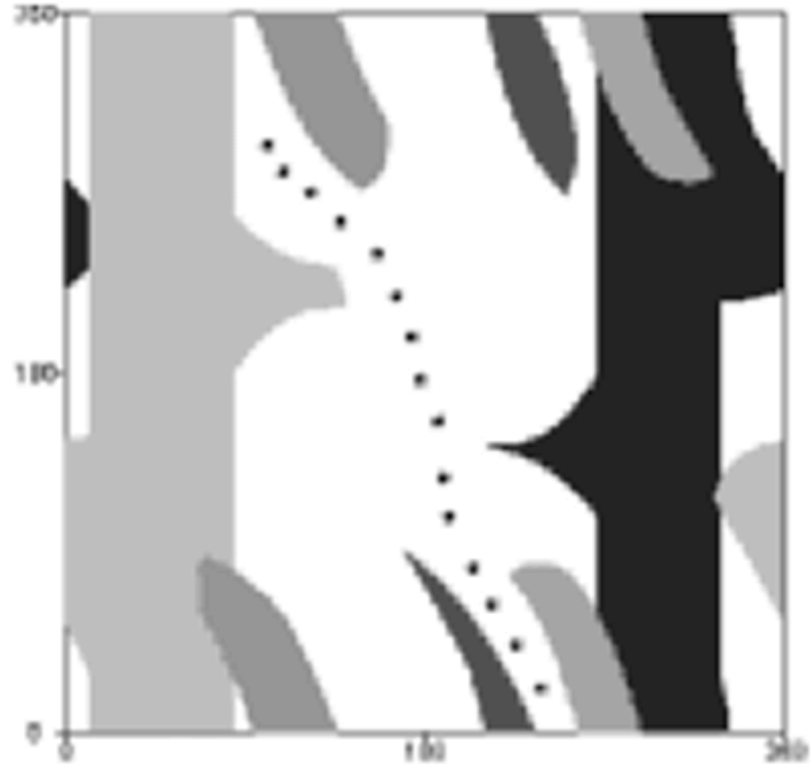


Example: Two-Joint Planar Arm





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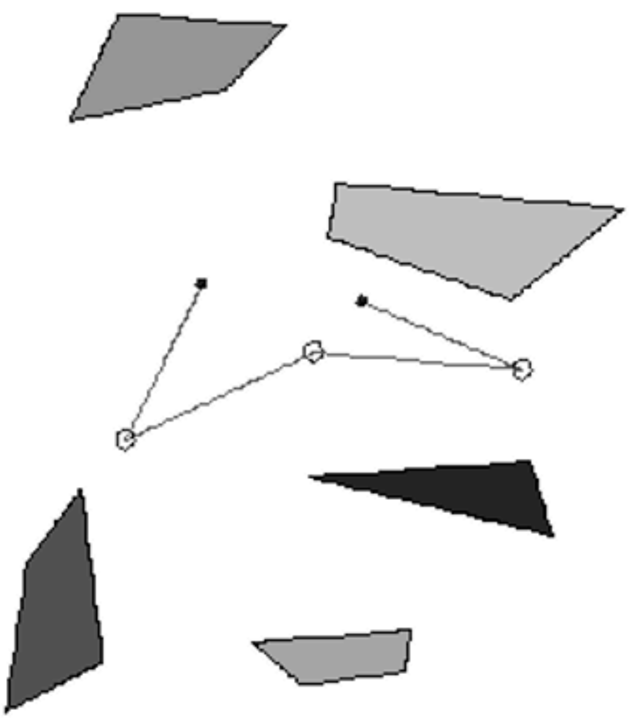

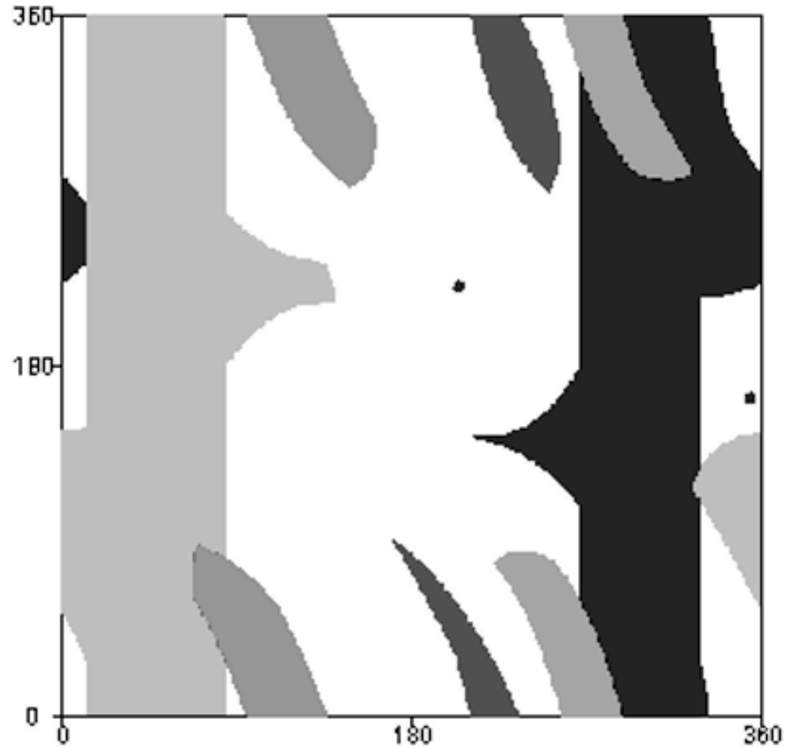

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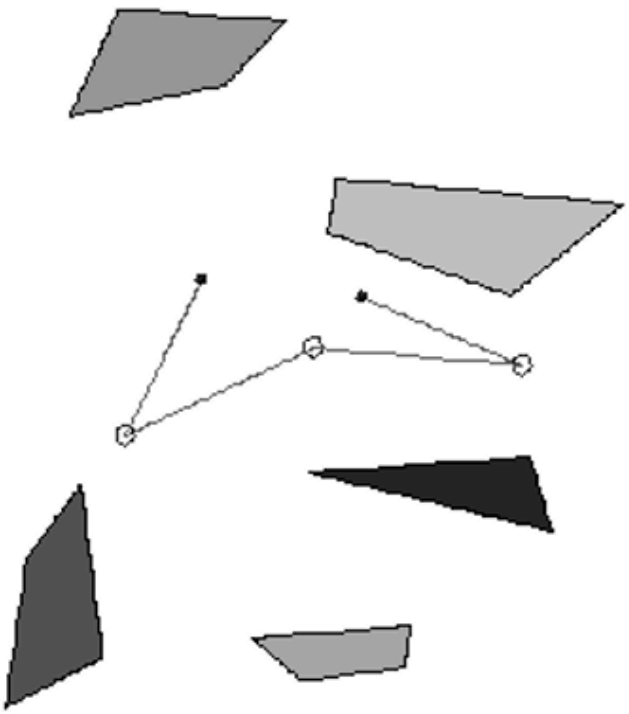

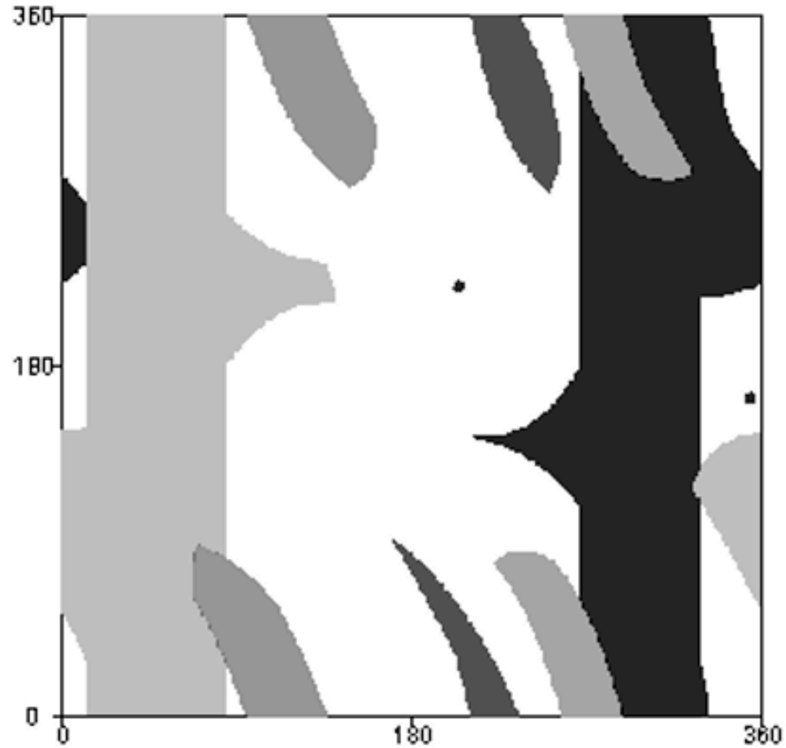

Definition

- *Degrees of Freedom*: the number of independent parameters that define its configuration or state. It is the *dimension* of the configuration space.

Degrees of Freedom

- Suppose that a robot can translate at a plane, what are its degrees of freedom?

Degrees of Freedom

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- If we have three planar robots?

Degrees of Freedom

- Suppose that a robot can translate at a plane, what are its degrees of freedom?
- If we have three planar robots?
- If we have three planar robots that can translate and rotate?

Degrees of Freedom

- What if have a two-link arm?

Degrees of Freedom

- What if have a two-link arm?

- What if $q_1 = q_2$?

Holonomic Constraint

- A constraint expressed as a function of the robot's configuration.

$$g(q, t) = 0$$

Degrees of Freedom

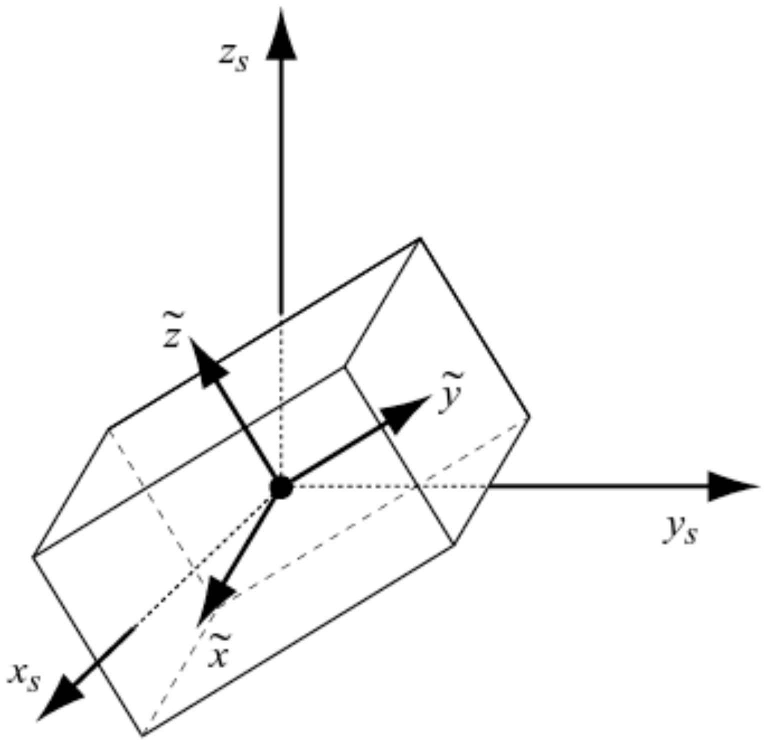
- We have a system of n coordinates subject to *m independent holonomic* constraints, we can achieve an $n - m$ dimensional configuration space.

Representations

- Sometimes it is beneficial to use representations with "extra" numbers, subject to specific constraints.

Orientation

$$R = \begin{bmatrix} \tilde{x}_1 & \tilde{y}_1 & \tilde{z}_1 \\ \tilde{x}_2 & \tilde{y}_2 & \tilde{z}_2 \\ \tilde{x}_3 & \tilde{y}_3 & \tilde{z}_3 \end{bmatrix}$$



Orientation

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- $||x|| = ||y|| = ||z|| = 1$

- $x^T y = y^T z = z^T x = 0$

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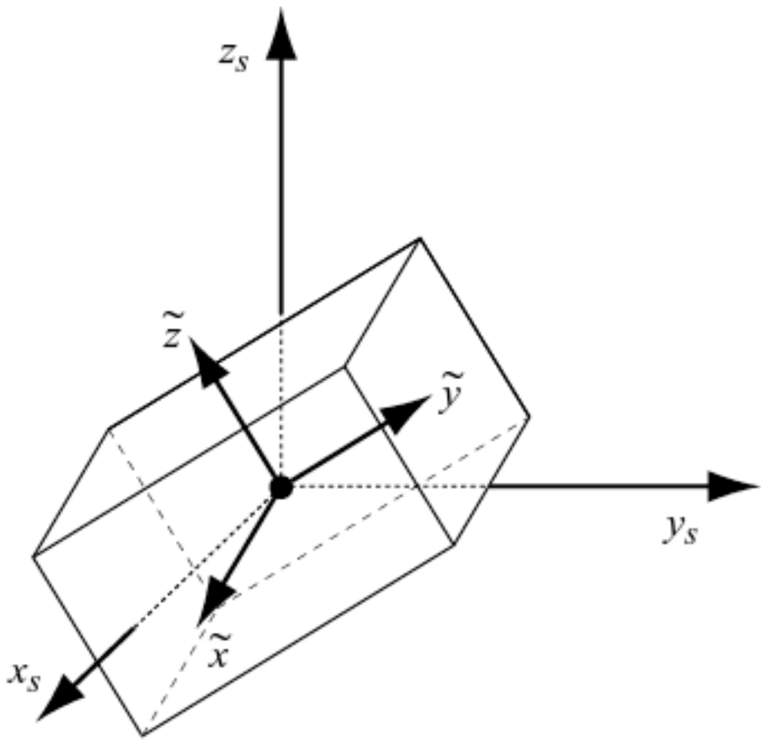

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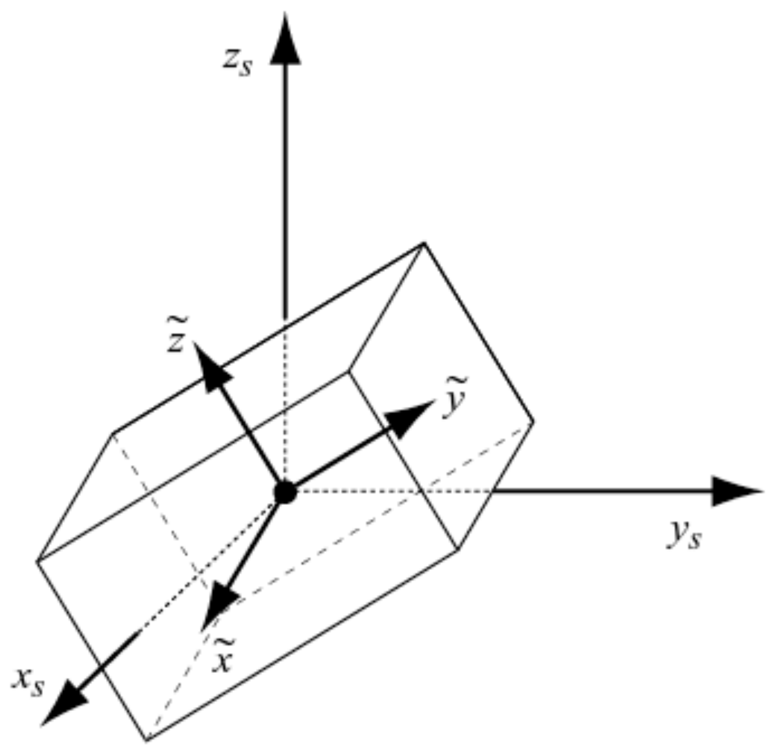
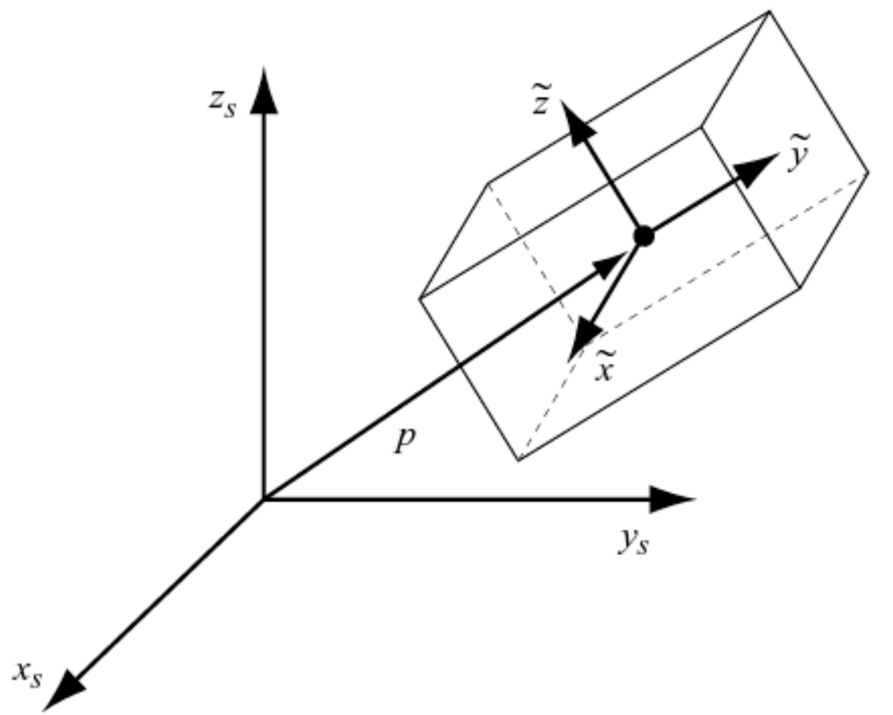


Planar Case

$$R = \begin{bmatrix} \tilde{x}_1 & \tilde{y}_1 \\ \tilde{x}_2 & \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \in SO(2)$$

Position and Orientation



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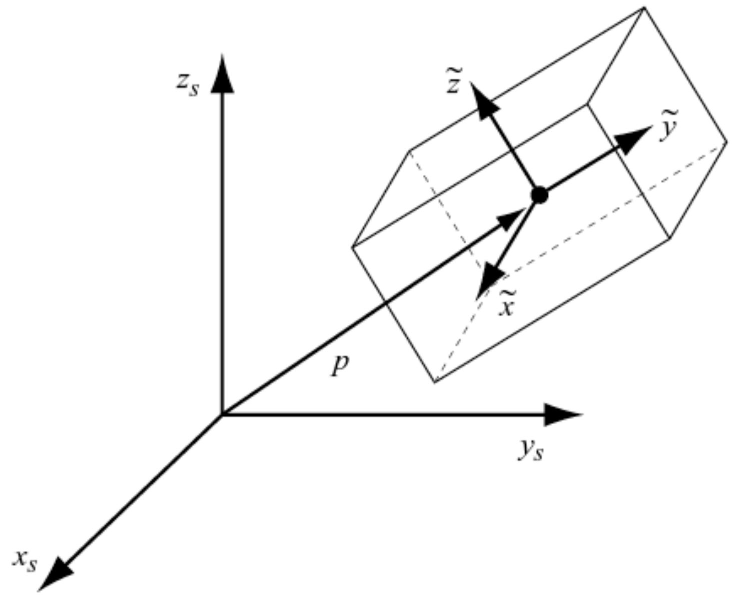


$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in SE(3), R \in SO(3), p \in \mathbb{R}^3$$

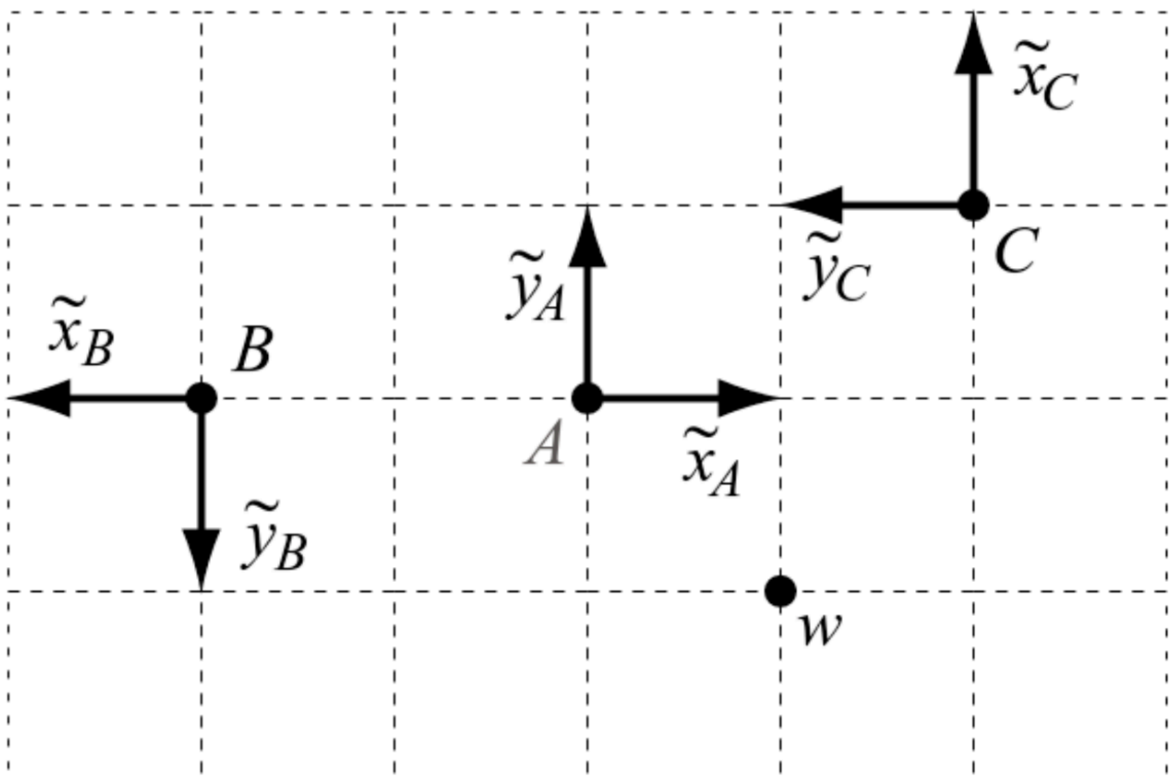
Uses of Matrix Representations

- The matrix groups $SO(n)$ and $SE(n)$ can be used to:
  - Represent rigid body configurations
  - change the reference frame for representing a configuration or a point
  - displace (move) a configuration or a point

Uses of Matrix Representations

- When the matrix is used for representing a configuration, we often call it a *frame*.
- When it is used for displacement or coordinate change, we often call it a *transform*.

Changing Frame of Reference



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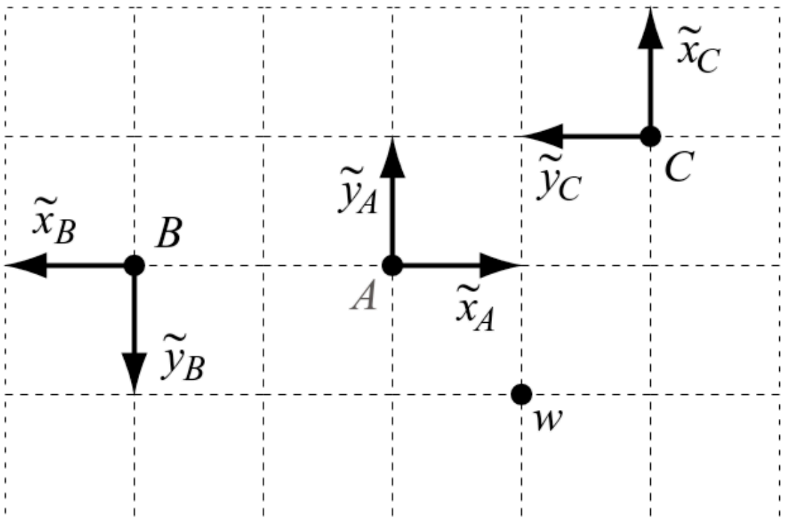

$$T_{AB} = \begin{bmatrix} R_{AB} & P_{AB} \\ 0_{1x3} & 1 \end{bmatrix}; R_{AB} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; P_{AB} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

$$T_{AB} = \begin{bmatrix} R_{AB} & P_{AB} \\ 0_{1x3} & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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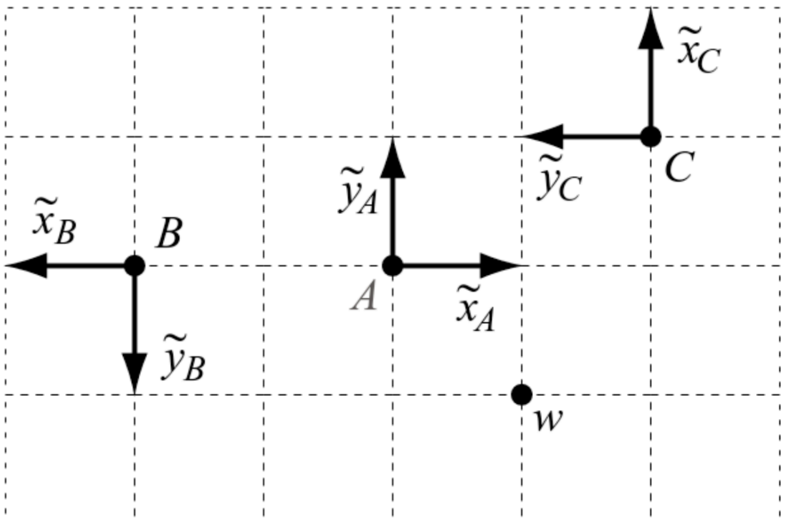

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$$R_{BC}?$$

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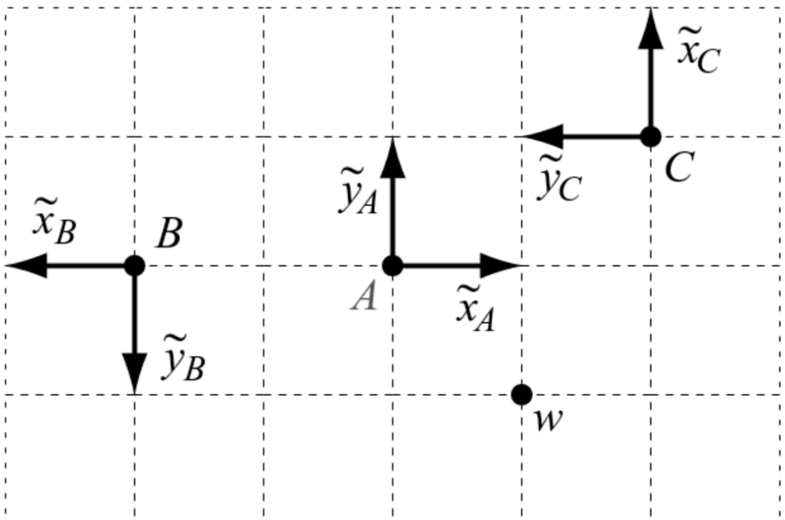

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$$R_{BC} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; P_{BC} = \begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix}$$

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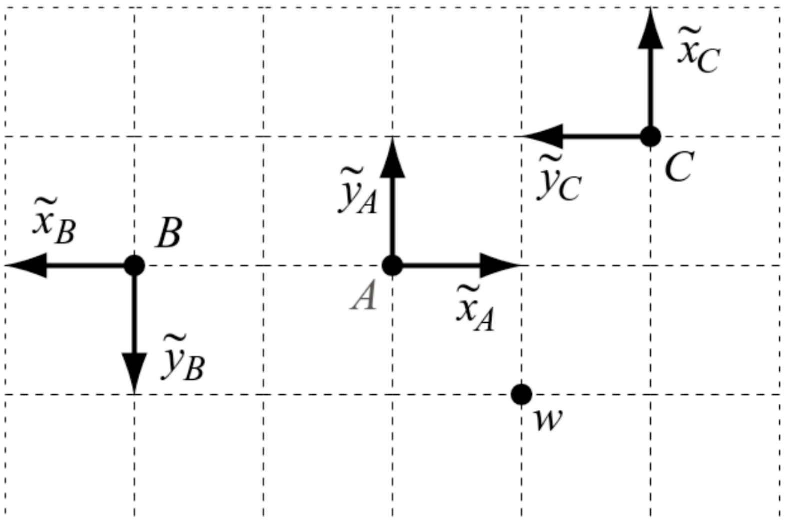

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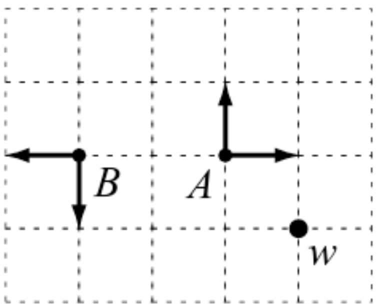
$$T_{AC} = T_{AB} * T_{BC} = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

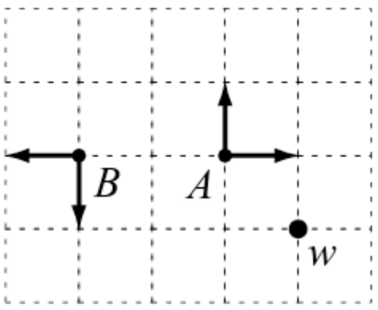
Changing Frame of Reference



$$w_B = [-3, 1, 0, 1]^T$$

$$w_A = ?$$

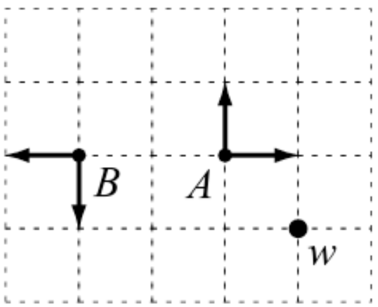
Changing Frame of Reference



$$w_B = [-3, 1, 0, 1]^T$$

$$w_A = [1, -1, 0, 1]^T$$

Changing Frame of Reference



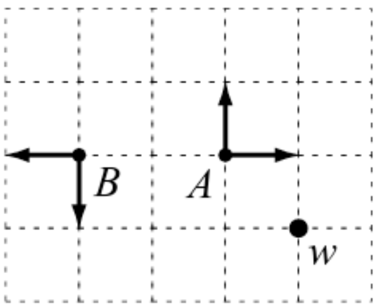
$$w_B = [-3, 1, 0, 1]^T$$

$$w_A = T_{AB} w_B$$

$$= \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [-3, 1, 0, 1]^T$$

$$= [1, -1, 0, 1]^T$$

Displacing a Point



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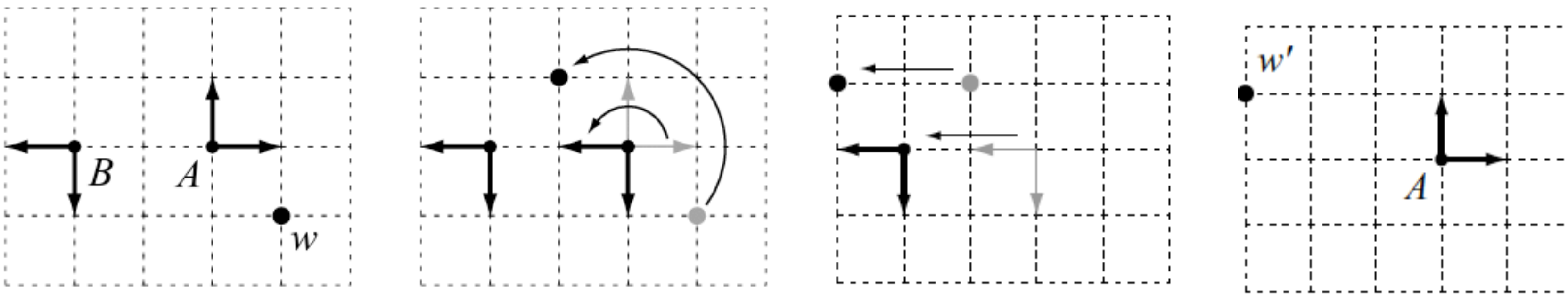

$$w_A = [1, -1, 0, 1]^T \qquad w'_A = T_{AB} w_A = [-3, 1, 0, 1]^T.$$

$$T_{AB} = \begin{bmatrix} R_{AB} & P_{AB} \\ 0_{1x3} & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Body-Frame Transformation

$$T_{AB'} = T_{AB} T_1 = \begin{bmatrix} R_{AB} R_1 & R_{AB} p_1 + p_{AB} \\ 0 & 1 \end{bmatrix}$$

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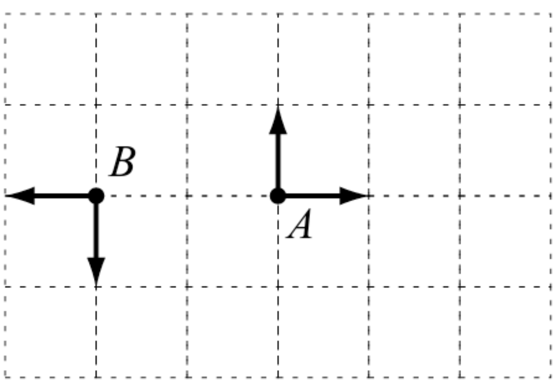

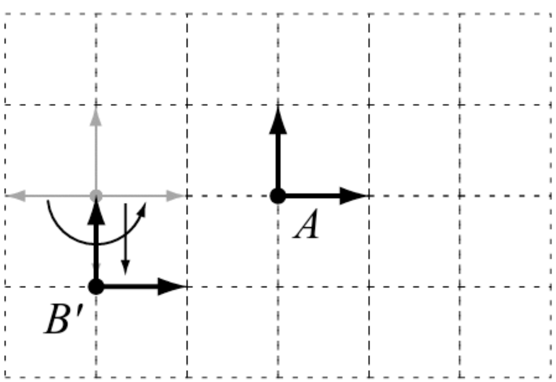

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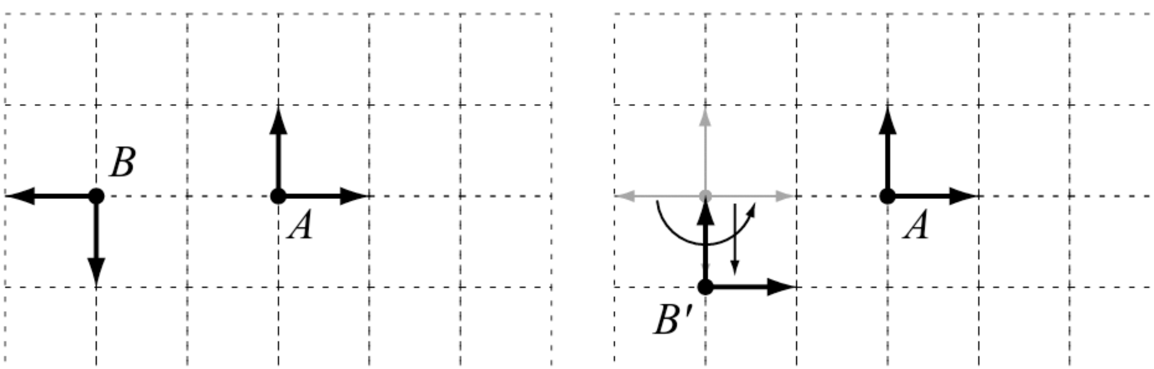
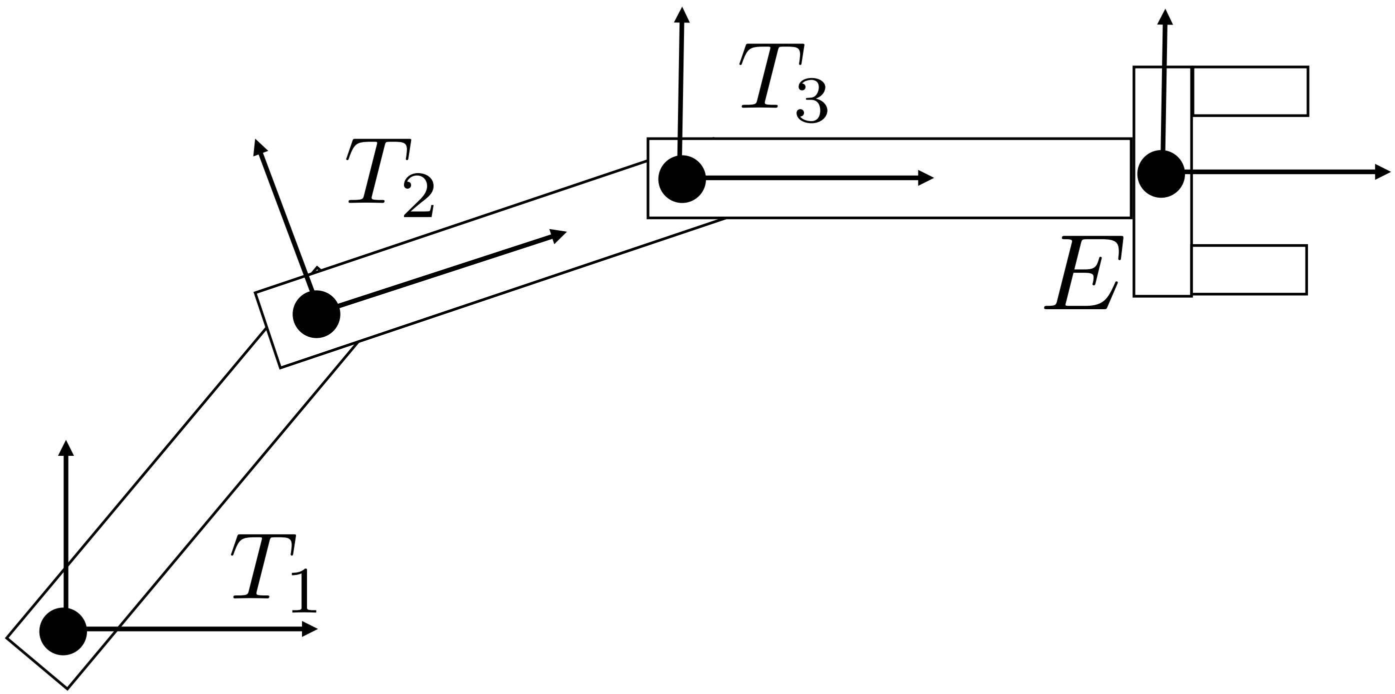
$$\begin{aligned} T_{AB'} &= T_{AB}T_1 \\ &= \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$



Example: Change Frame of Reference



$$T'_E = T_1 T_2 T_3$$

Example Configuration Spaces

- Mobile robot translating on a plane:
- Mobile robot translating and rotating on a plane:
- A spacecraft:
- An n-joint revolute arm:
- A mobile robot with a two-joint arm

Example Configuration Spaces

- Mobile robot translating on a plane: $\mathbb{R}^2$
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Example Configuration Spaces

- Mobile robot translating on a plane: $\mathbb{R}^2$
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- A spacecraft: $SE(3)$ or $\mathbb{R}^3 \times SO(3)$
- An n-joint revolute arm: T^n
- A mobile robot with a two-joint arm $SE(2) \times T^2$