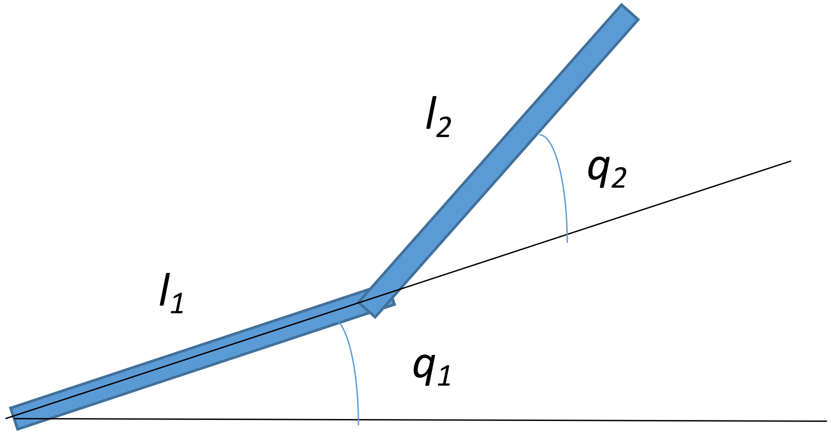
USC Viterbi
School of Engineering

Forward and Inverse Kinematics

CSCI 545 Introduction to Robotics
Instructor: Stefanos Nikolaidis

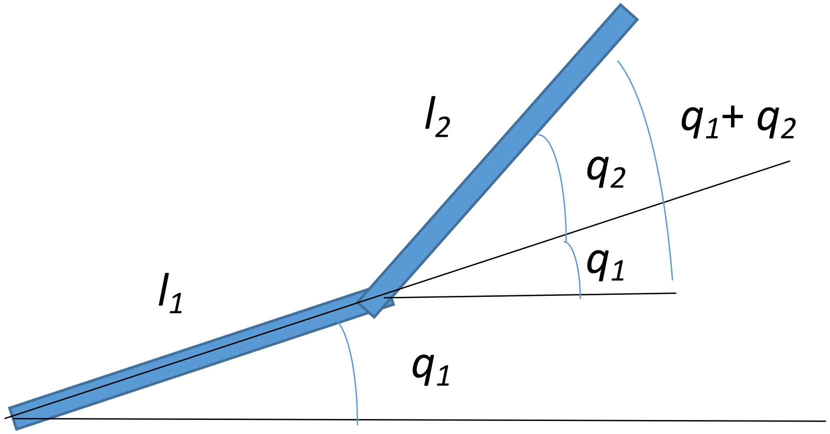
Resources : *16-811: Math Fundamentals for Robotics by Michael Erdmann, Carnegie Mellon University*

Forward Kinematics



$$\phi : Q \to \mathbb{R}^2$$

Forward Kinematics



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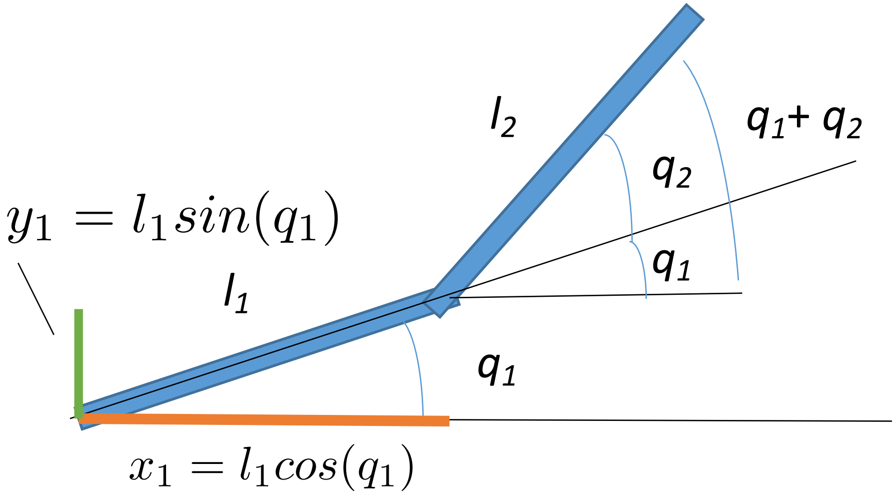
Forward Kinematics



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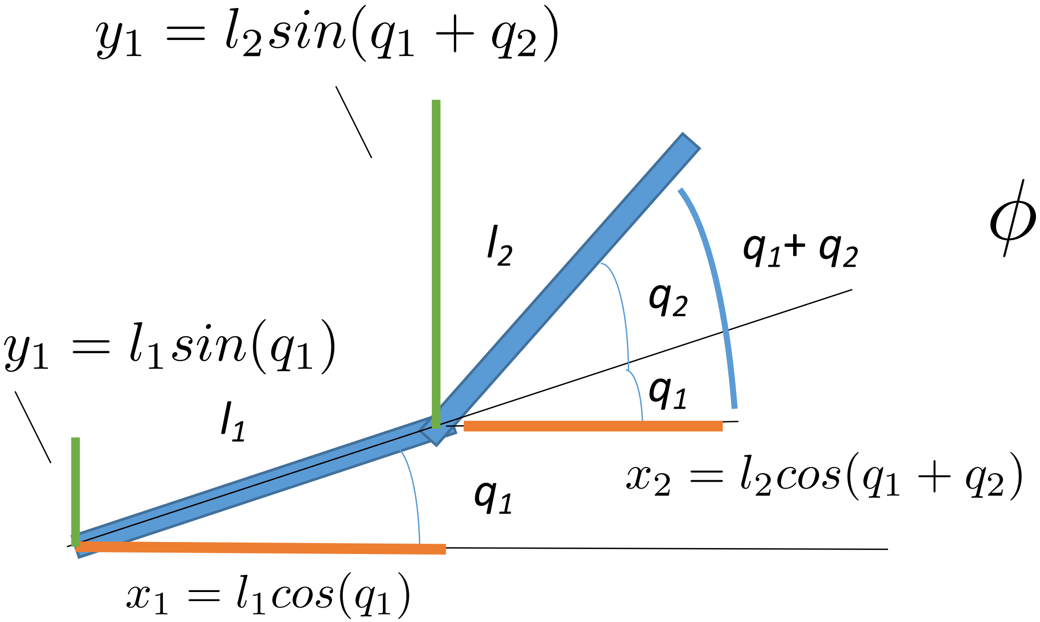
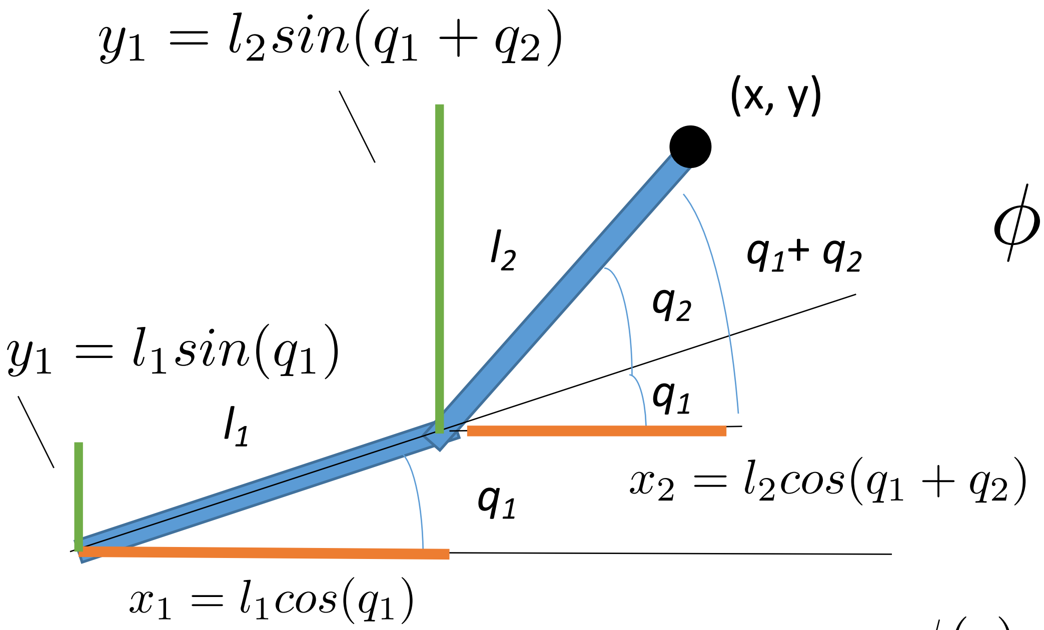
Forward Kinematics



$$\phi : Q \rightarrow \mathbb{R}^2$$

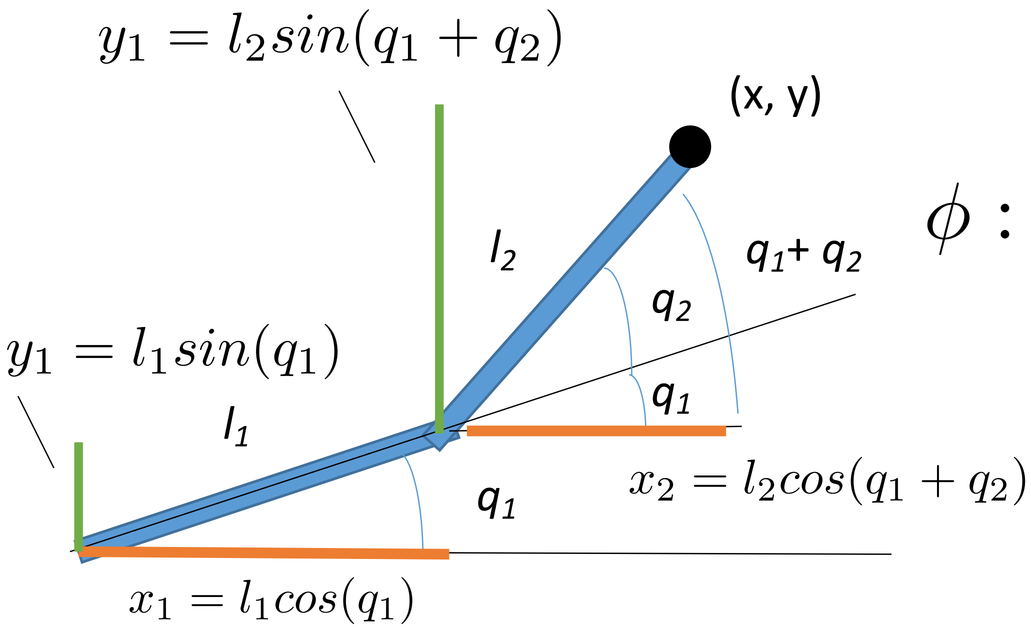
Forward Kinematics



$$\phi : Q \rightarrow \mathbb{R}^2$$

$$\begin{aligned} \phi(q) &= \begin{bmatrix} \phi_1(q) \\ \phi_2(q) \end{bmatrix} \\ &= \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \\ &= \begin{bmatrix} l_1 cos(q_1) + l_2 cos(q_1 + q_2) \\ l_1 sin(q_1) + l_2 sin(q_1 + q_2) \end{bmatrix} \end{aligned}$$

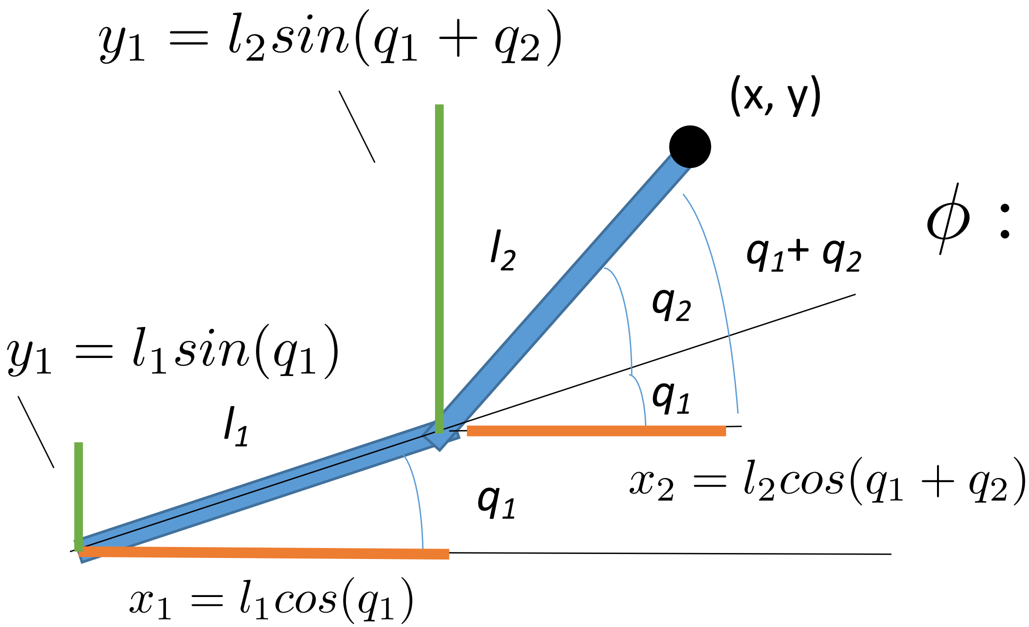
Forward Kinematics



$$\phi : Q \rightarrow SE(2)$$

$$T_1 = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 & l_1 \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics

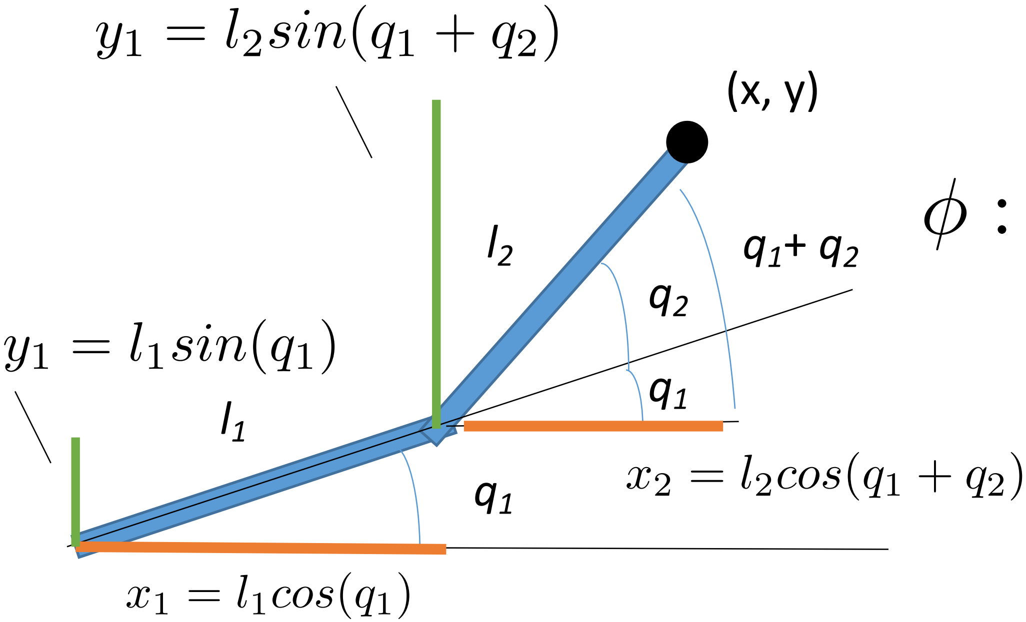


$$\phi : Q \rightarrow SE(2)$$

$$T_1 = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 & l_1 \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 & l_2 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & l_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics

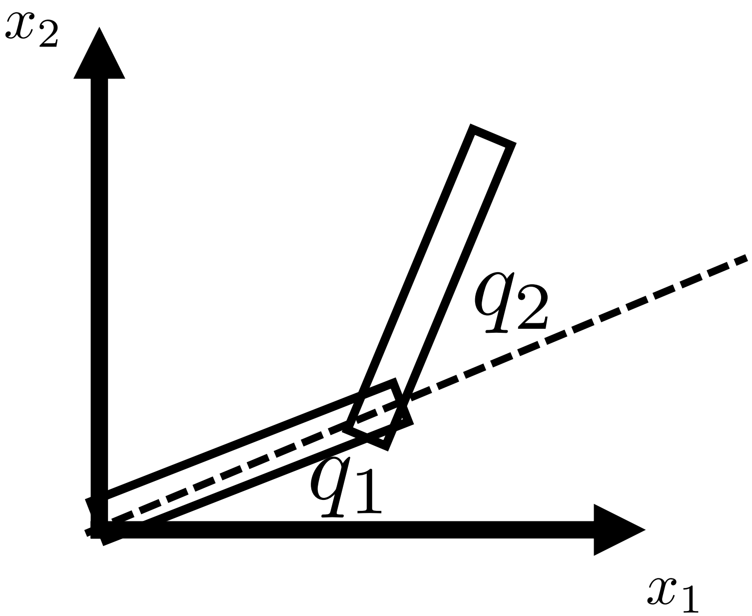


$$\phi : Q \rightarrow SE(2)$$

$$cos(q_1 + q_2)$$

$$T_1 * T_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

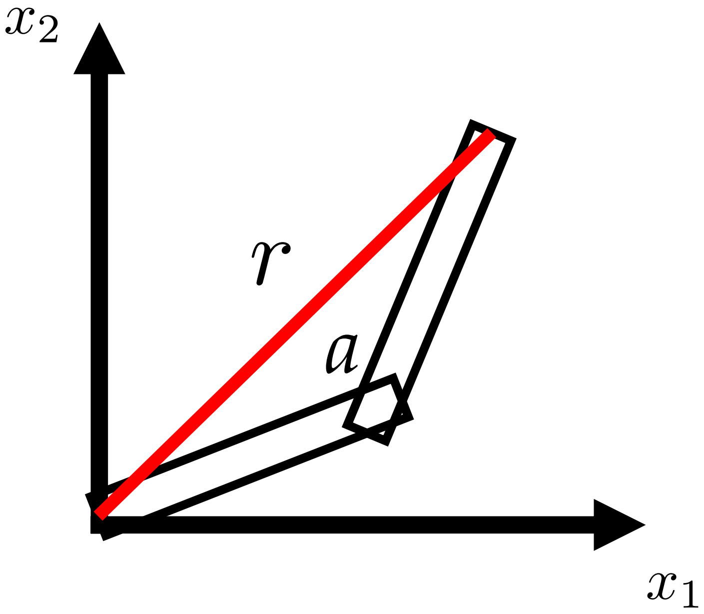
Inverse Kinematics



$$\phi^{-1} : \mathbb{R}^2 \to Q$$

We will express q_1, q_2 with respect to x_1, x_2

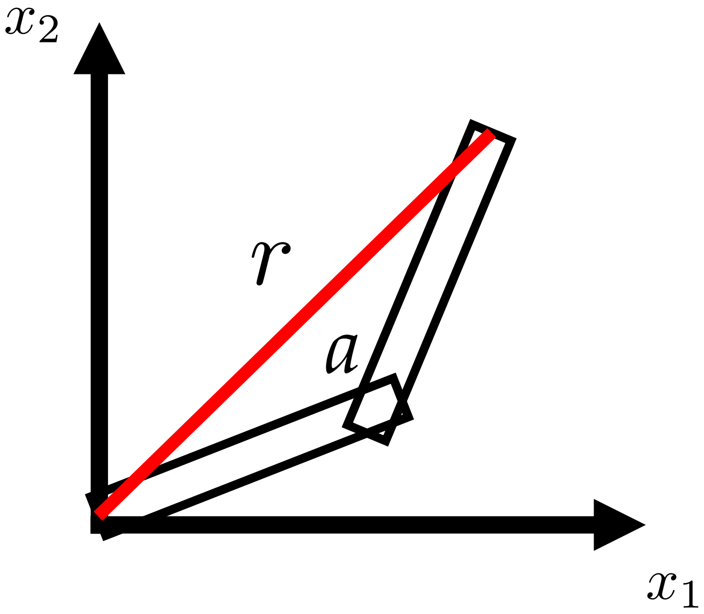
Inverse Kinematics



$$\phi^{-1} : \mathbb{R}^2 \to Q$$

$$r = \sqrt{x_1^2 + x_2^2}$$

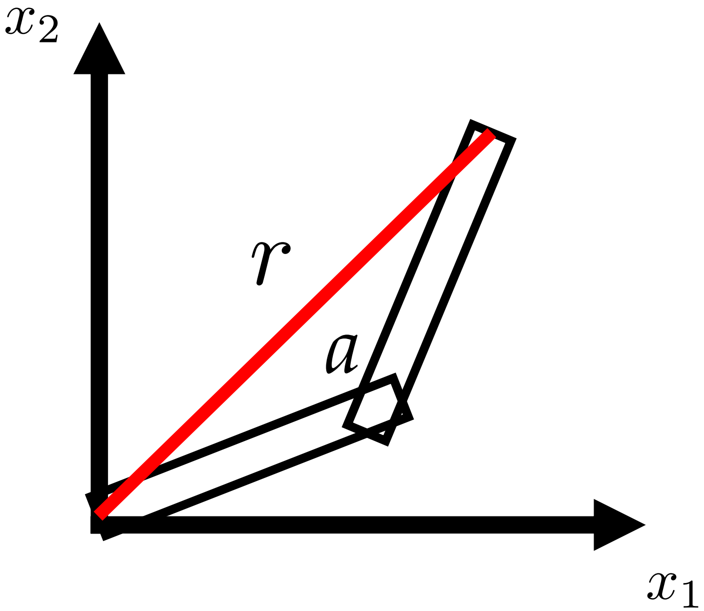
Inverse Kinematics



$$\phi^{-1} : \mathbb{R}^2 \to Q$$

$$r = \sqrt{l_1^2 + l_2^2 - 2 l_1 l_2 cos\alpha}$$

Inverse Kinematics

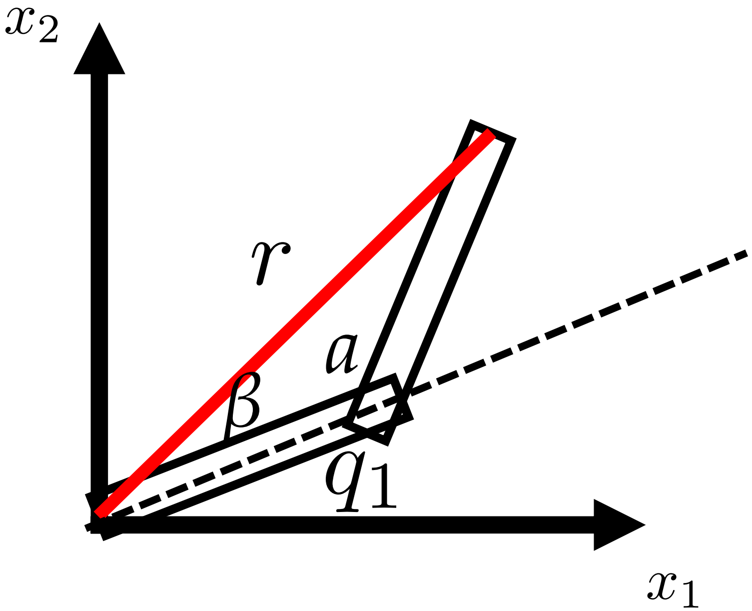


$$\phi^{-1} : \mathbb{R}^2 \rightarrow Q$$

$$r = \sqrt{l_1^2 + l_2^2 - 2l_1 l_2 cos\alpha}$$

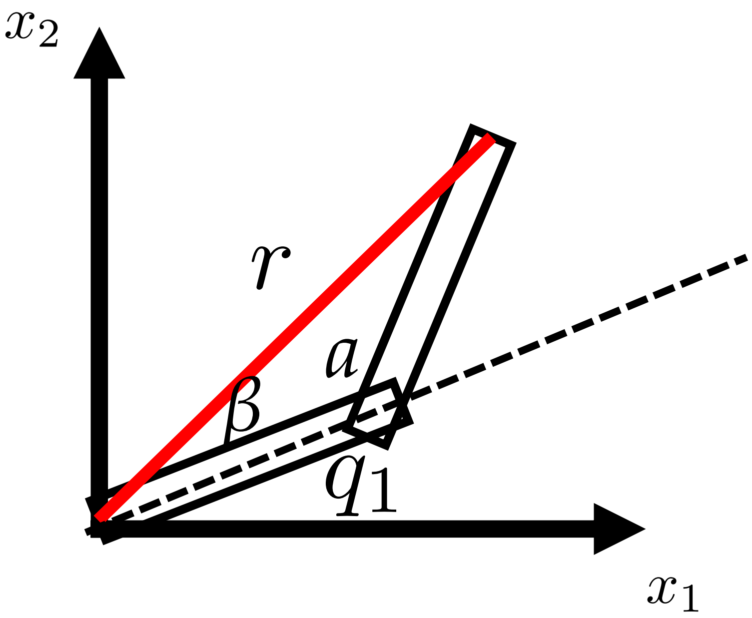
$$\alpha = cos^{-1} \frac{r^2 - l_1^2 - l_2^2}{-2l_1 l_2}$$

Inverse Kinematics



$$\phi^{-1} : \mathbb{R}^2 \to Q$$

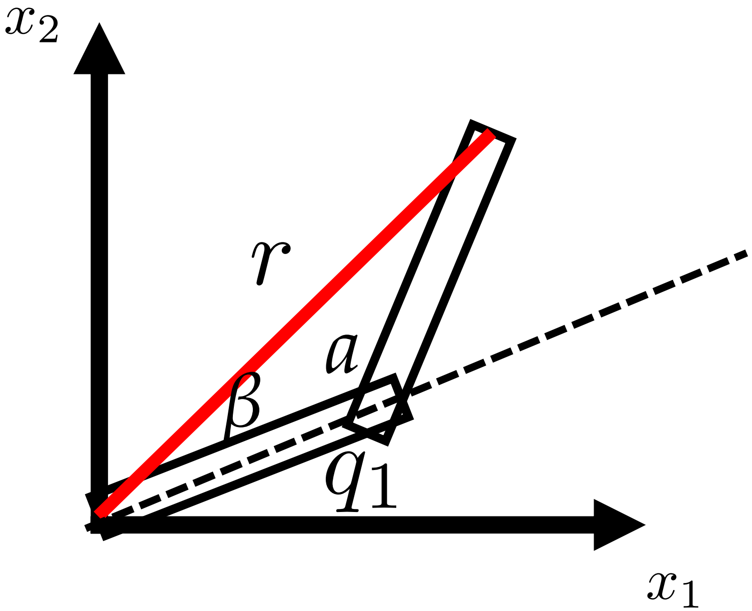
Inverse Kinematics



$$\phi^{-1} : \mathbb{R}^2 \to Q$$

$$q_1 = \text{atan2}(y, x) - \beta$$

Inverse Kinematics

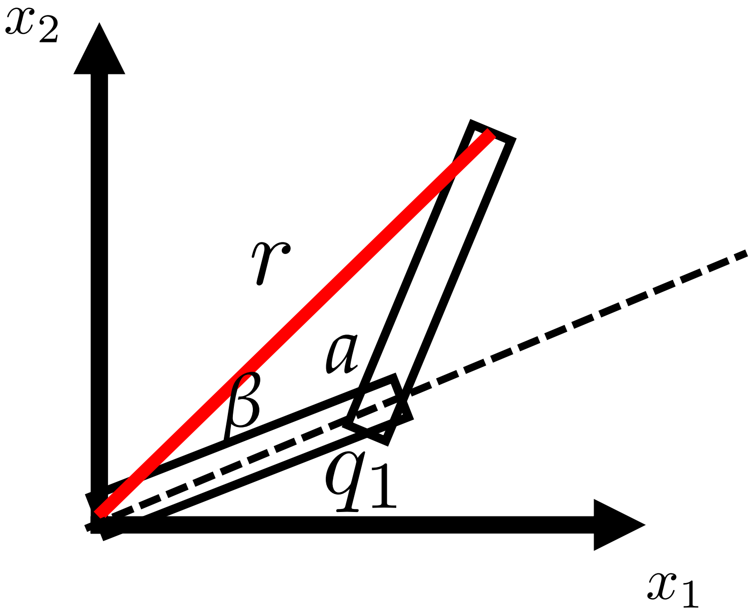


$$\phi^{-1} : \mathbb{R}^2 \rightarrow Q$$

$$q_1 = \text{atan2}(y, x) - \beta$$

$$\beta = cos^{-1} \frac{l_2^2 - l_1^2 - r^2}{-2l_1 r}$$

Inverse Kinematics

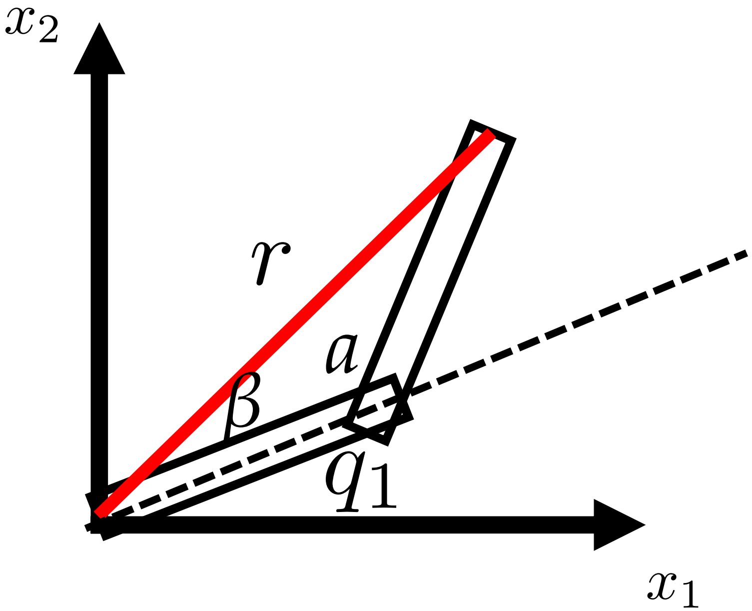


$$\phi^{-1} : \mathbb{R}^2 \to Q$$

$$\alpha = cos^{-1} \frac{r^2 - l_1^2 - l_2^2}{-2l_1 l_2}$$

$$\beta = cos^{-1} \frac{l_2^2 - l_1^2 - r^2}{-2l_1 r}$$

Inverse Kinematics



$$\phi^{-1} : \mathbb{R}^2 \to Q$$

$$\alpha = cos^{-1} \frac{r^2 - l_1^2 - l_2^2}{-2 l_1 l_2}$$

What if $r > l_1 + l_2$?

Inverse Kinematics

$$r = \sqrt{l_1^2 + l_2^2 - 2l_1 l_2 cos\alpha}$$

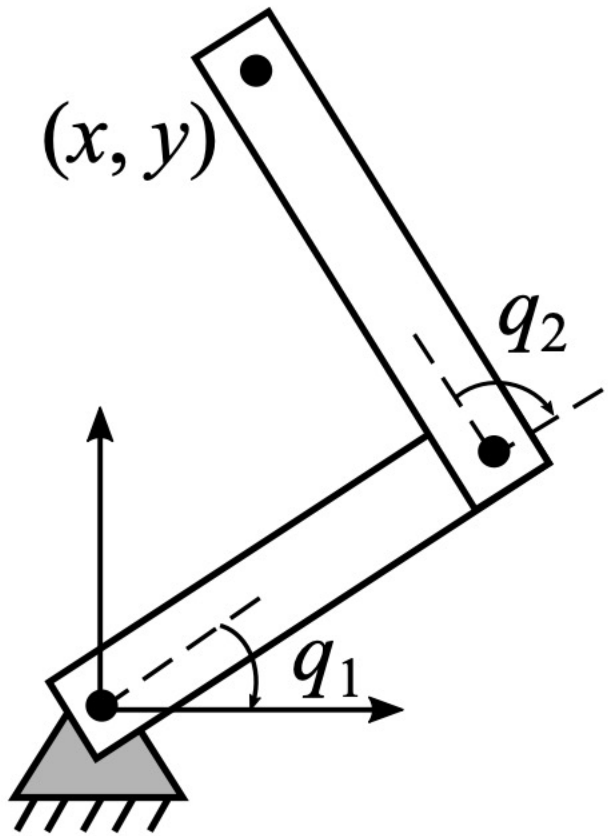
$$\alpha = cos^{-1}\frac{r^2 - l_1^2 - l_2^2}{-2l_1 l_2}$$

$$= cos^{-1}\frac{r^2 - (l_1 + l_2)^2 + 2l_1 l_2}{-2l_1 l_2}$$

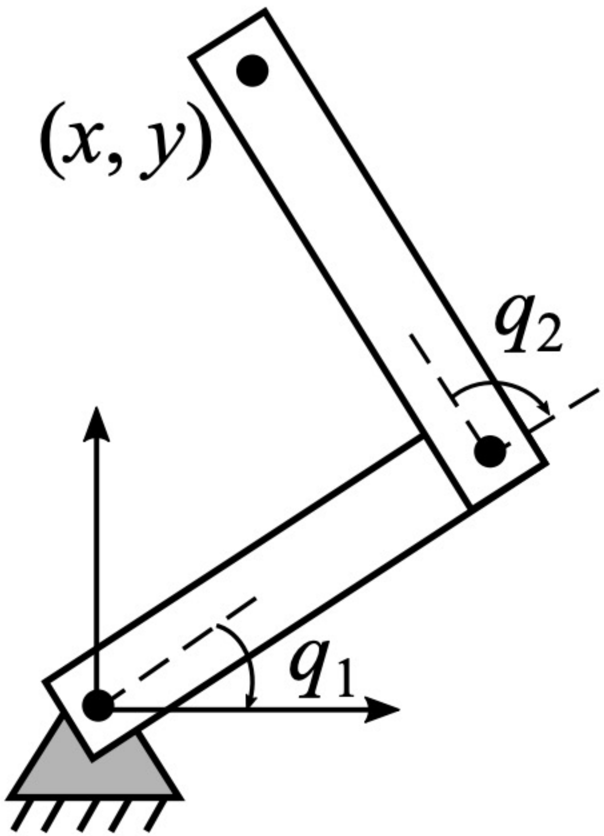
$$cosa = \frac{r^2 - (l_1 + l_2)^2 + 2l_1 l_2}{-2l_1 l_2}$$

$$cosa = \frac{r^2 - (l_1 + l_2)^2}{-2l_1 l_2} - 1$$

Derivation of Jacobian Equation

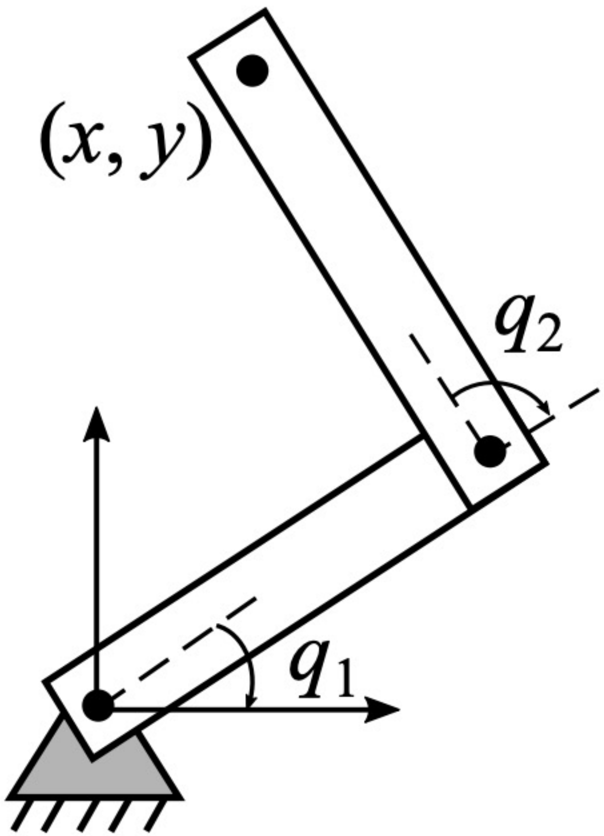


Derivation of Jacobian Equation



$$p = \phi(q)$$

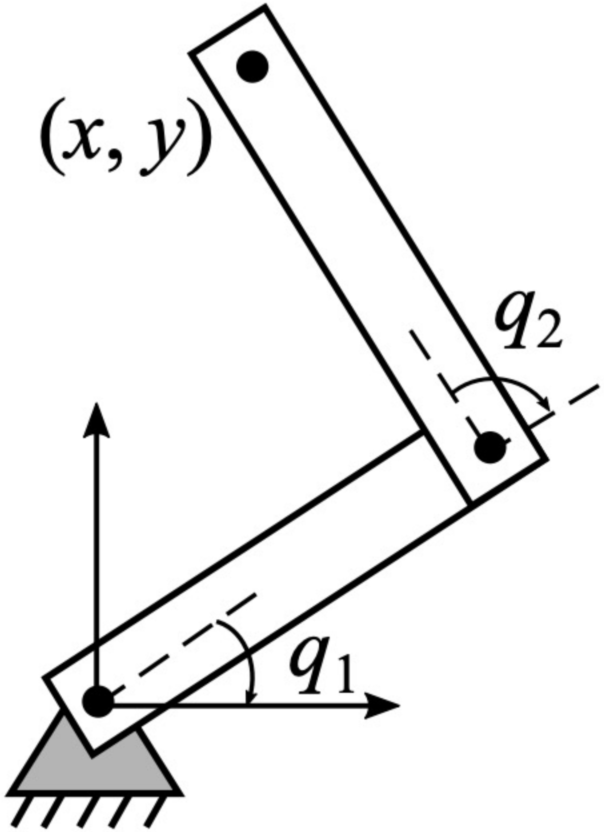
Derivation of Jacobian Equation



$$p = \phi(q)$$

$$\frac{dp}{dt} = \nabla_q \phi \ \frac{dq}{dt} \Rightarrow$$

Derivation of Jacobian Equation

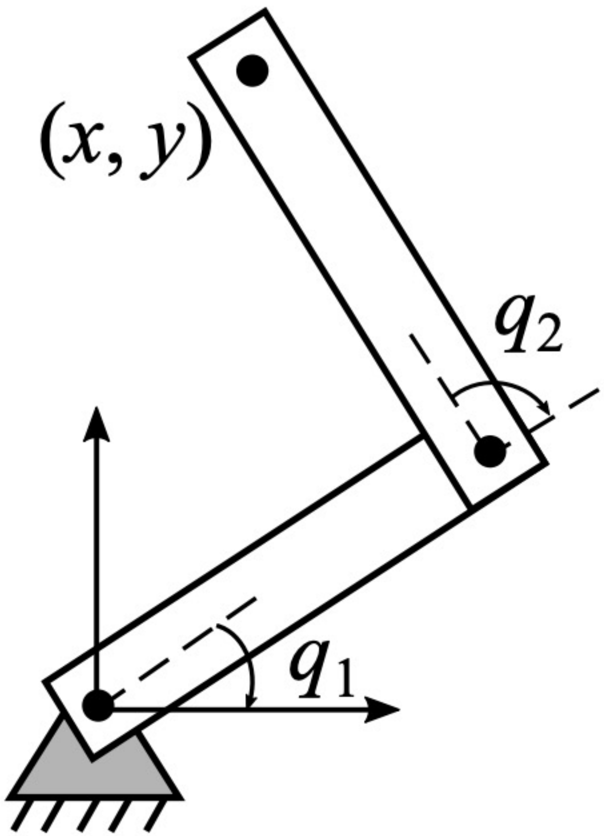


$$p = \phi(q)$$

$$\frac{dp}{dt} = \nabla_q \phi \; \frac{dq}{dt} \Rightarrow$$

$$\dot{p} = \nabla_q \phi \; \dot{q}$$

Derivation of Jacobian Equation



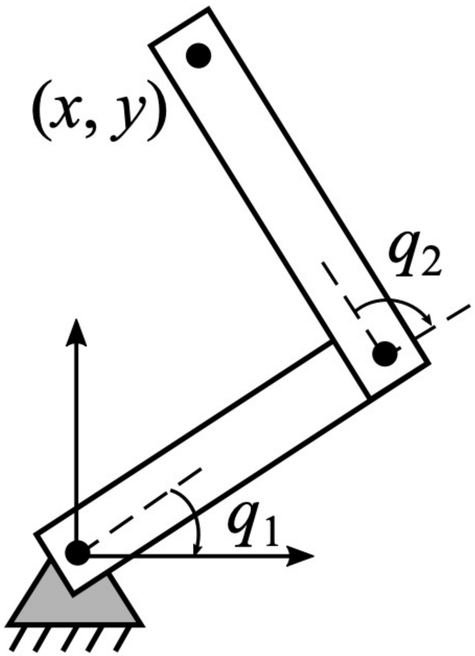
$$p = \phi(q)$$

$$\frac{dp}{dt} = \nabla_q \phi \ \frac{dq}{dt} \Rightarrow$$

$$\dot{p} = \nabla_q \phi \ \dot{q}$$

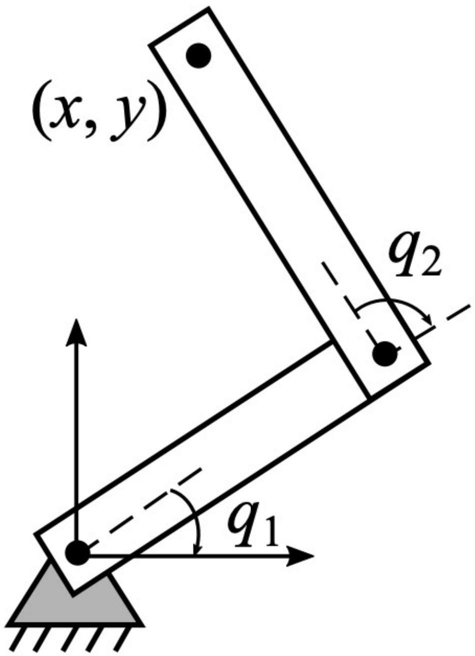
$$\dot{p} = J \dot{q}$$

Example: 2-R Planar Manipulator



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 cos(q_1) + l_2 cos(q_1 + q_2) \\ l_1 sin(q_1) + l_2 sin(q_1 + q_2) \end{bmatrix}$$

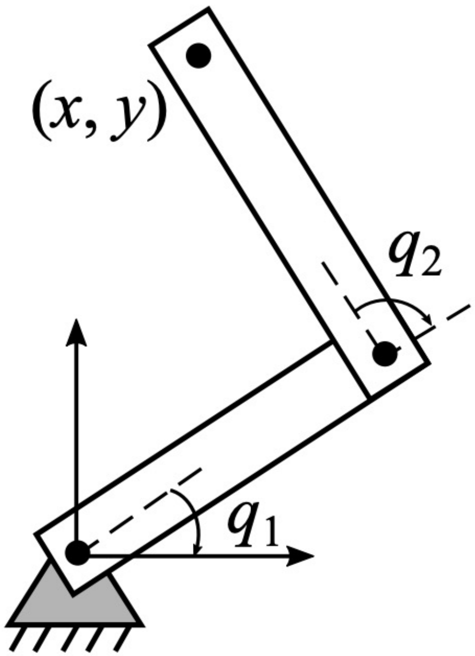
Example: 2-R Planar Manipulator



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 cos(q_1) + l_2 cos(q_1 + q_2) \\ l_1 sin(q_1) + l_2 sin(q_1 + q_2) \end{bmatrix}$$

$$= \phi(q_1, q_2)$$

Example: 2-R Planar Manipulator

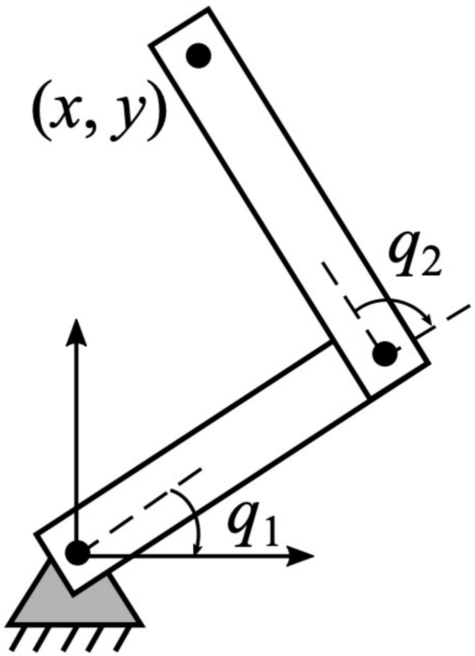


$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 cos(q_1) + l_2 cos(q_1 + q_2) \\ l_1 sin(q_1) + l_2 sin(q_1 + q_2) \end{bmatrix}$$

$$= \phi(q_1, q_2)$$

$$= \begin{bmatrix} \phi_1(q_1, q_2) \\ \phi_2(q_1, q_2) \end{bmatrix}$$

Example: 2-R Planar Manipulator

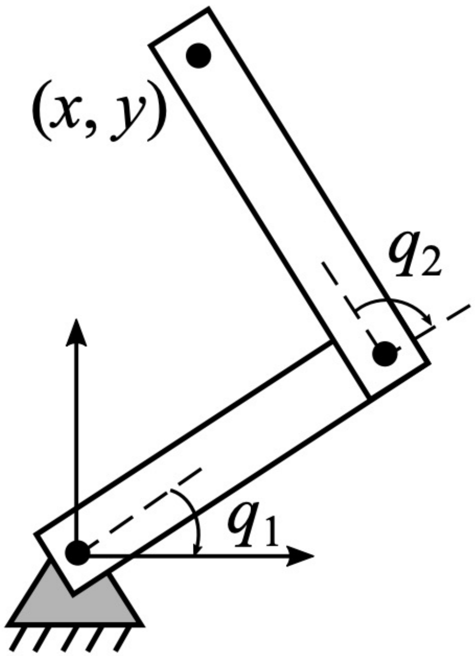


$$J = \nabla_q \phi = \begin{bmatrix} \frac{\partial \phi_1}{\partial q_1} & \frac{\partial \phi_1}{\partial q_2} \\ \frac{\partial \phi_2}{\partial q_1} & \frac{\partial \phi_2}{\partial q_2} \end{bmatrix}$$

Example: 2-R Planar Manipulator

$$
\begin{aligned}
J &= \nabla_q \phi \\
&= \begin{bmatrix} \frac{\partial \phi_1}{\partial q_1} & \frac{\partial \phi_1}{\partial q_2} \\ \frac{\partial \phi_2}{\partial q_1} & \frac{\partial \phi_2}{\partial q_2} \end{bmatrix} \\
&= \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}
\end{aligned}
$$

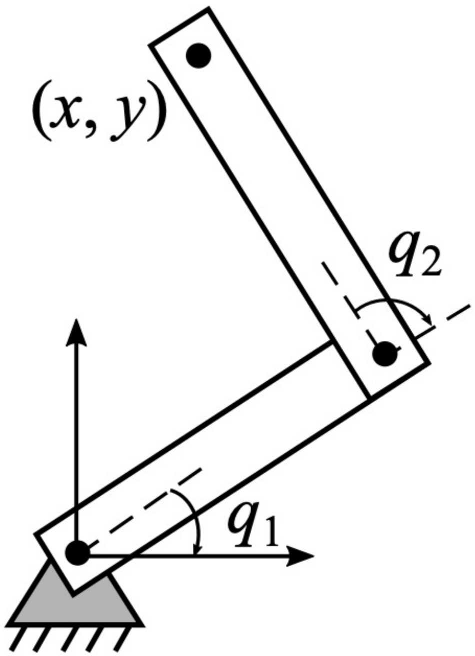
Example: 2-R Planar Manipulator



$$l_1 = l_2 = 1, q_1 = \frac{\pi}{4}, q_2 = \frac{\pi}{2}$$

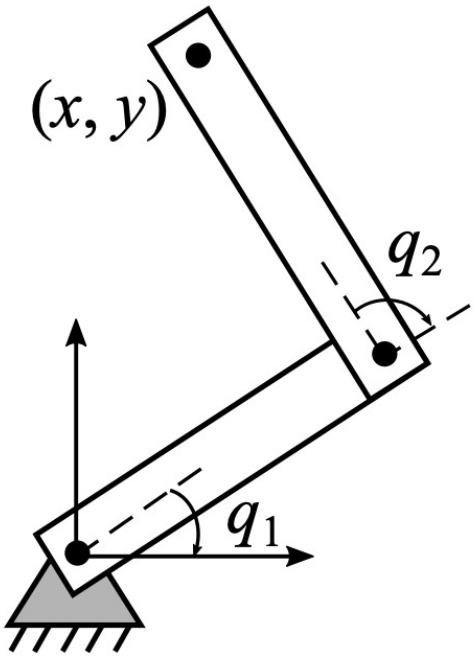
$$\dot{q}_1 = 1, \dot{q}_2 = 0$$

Example: 2-R Planar Manipulator



$$\dot{p} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

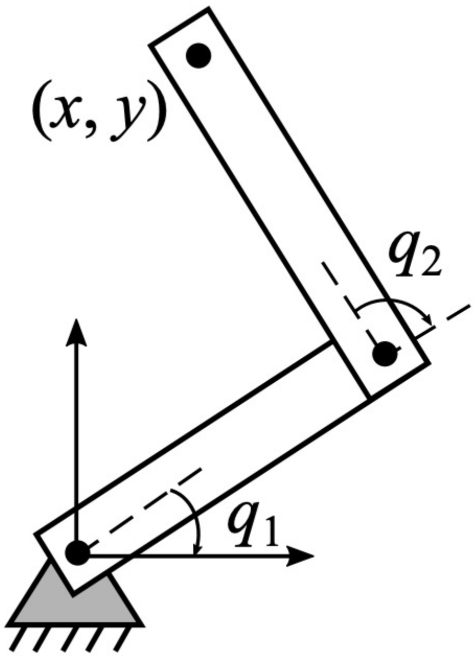
Example: 2-R Planar Manipulator



$$\dot{p} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$= J\dot{q}$$

Example: 2-R Planar Manipulator

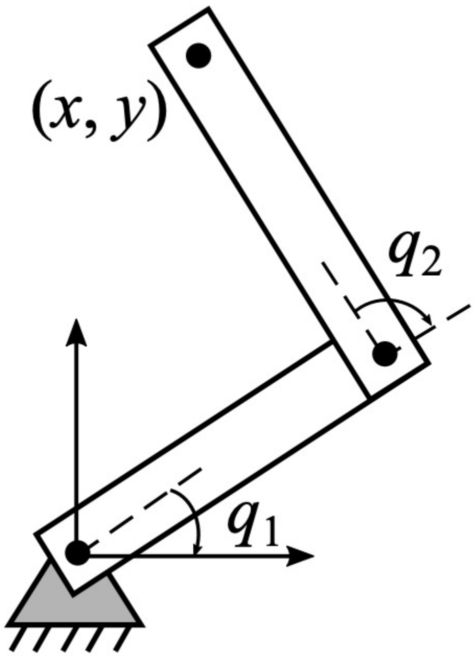


$$\dot{p} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$= J\dot{q}$$

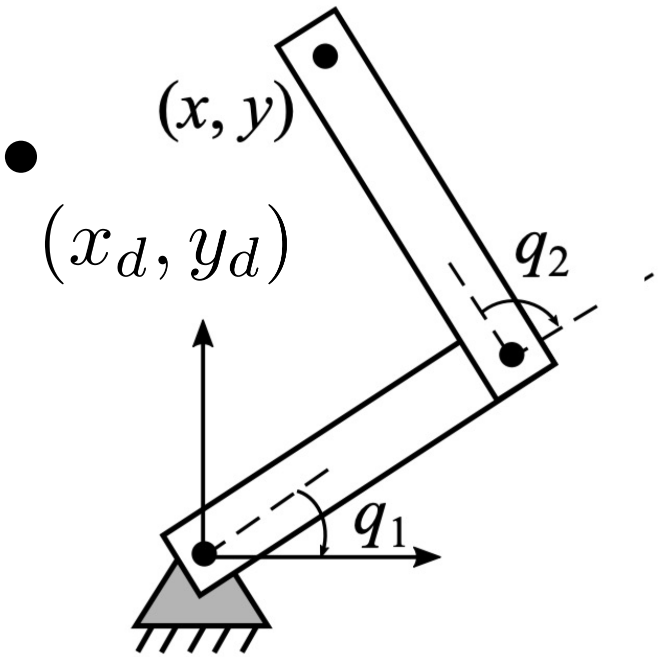
$$= \begin{bmatrix} -\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example: 2-R Planar Manipulator



$$
\begin{aligned}
\dot{p} &= \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \\
&= J\dot{q} \\
&= \begin{bmatrix} -\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}
\end{aligned}
$$

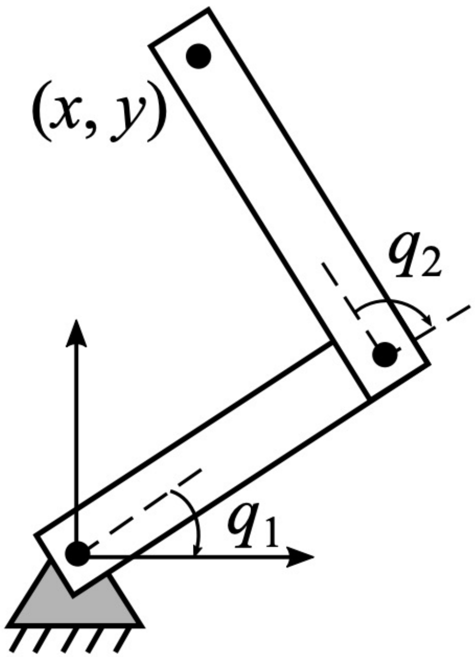
Jacobian Inverse?



$$\dot{p} = J\dot{q}$$

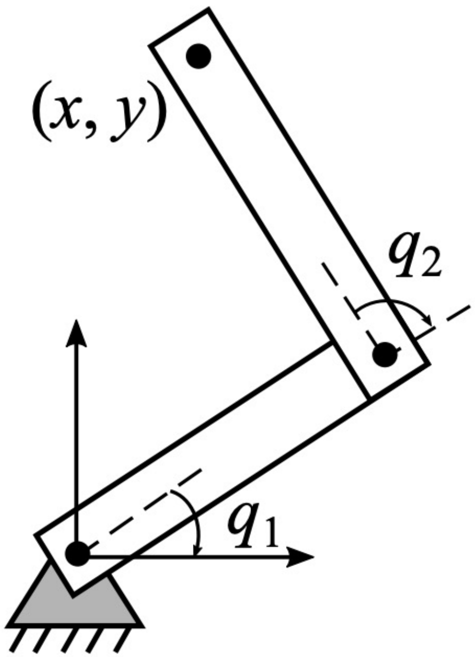
Why not solve for the joint velocities?

When is Jacobian non-invertible?



When is Jacobian non-invertible?

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$



When is Jacobian non-invertible?

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$

$$\det(J) = 0$$

When is Jacobian non-invertible?

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$$\det(J) = 0$$

$$-l_1 s_1 l_2 c_{12} - l_2^2 s_{12} c_{12} + l_1 l_2 c_1 s_{12} + l_2^2 s_{12} c_{12} = 0$$

When is Jacobian non-invertible?

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$

$$\det(J) = 0$$

$$-l_1 s_1 l_2 c_{12} - l_2^2 s_{12} c_{12} + l_1 l_2 c_1 s_{12} + l_2^2 s_{12} c_{12} = 0$$

$$-l_1 l_2 s_1 c_{12} + l_1 l_2 c_1 s_{12} = 0$$

When is Jacobian non-invertible?

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$

$$\det(J) = 0$$

$$-l_1 s_1 l_2 c_{12} - l_2^2 s_{12} c_{12} + l_1 l_2 c_1 s_{12} + l_2^2 s_{12} c_{12} = 0$$

$$-l_1 l_2 s_1 c_{12} + l_1 l_2 c_1 s_{12} = 0$$

$$c_1 s_{12} - s_1 c_{12} = 0$$

When is Jacobian non-invertible?

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$

$$\det(J) = 0$$

$$-l_1 s_1 l_2 c_{12} - l_2^2 s_{12} c_{12} + l_1 l_2 c_1 s_{12} + l_2^2 s_{12} c_{12} = 0$$

$$-l_1 l_2 s_1 c_{12} + l_1 l_2 c_1 s_{12} = 0$$

$$c_1 s_{12} - s_1 c_{12} = 0$$

$$c_1(s_1 c_2 + c_1 s_2) - s_1(c_1 c_2 - s_1 s_2) = 0$$

When is Jacobian non-invertible?

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$

$$\begin{aligned} \det(J) &= 0 \\ -l_1 s_1 l_2 c_{12} - l_2^2 s_{12} c_{12} + l_1 l_2 c_1 s_{12} + l_2^2 s_{12} c_{12} &= 0 \\ -l_1 l_2 s_1 c_{12} + l_1 l_2 c_1 s_{12} &= 0 \\ c_1 s_{12} - s_1 c_{12} &= 0 \\ c_1 (s_1 c_2 + c_1 s_2) - s_1 (c_1 c_2 - s_1 s_2) &= 0 \\ s_1 c_1 c_2 + c_1^2 s_2 - s_1 c_1 c_2 + s_1^2 s_2 &= 0 \end{aligned}$$

When is Jacobian non-invertible?

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$

$$\det(J) = 0$$

$$-l_1 s_1 l_2 c_{12} - l_2^2 s_{12} c_{12} + l_1 l_2 c_1 s_{12} + l_2^2 s_{12} c_{12} = 0$$

$$-l_1 l_2 s_1 c_{12} + l_1 l_2 c_1 s_{12} = 0$$

$$c_1 s_{12} - s_1 c_{12} = 0$$

$$c_1 (s_1 c_2 + c_1 s_2) - s_1 (c_1 c_2 - s_1 s_2) = 0$$

$$s_1 c_1 c_2 + c_1^2 s_2 - s_1 c_1 c_2 + s_1^2 s_2 = 0$$

$$s_2 = 0$$

When is Jacobian non-invertible?

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$

$$\det(J) = 0$$

$$-l_1 s_1 l_2 c_{12} - l_2^2 s_{12} c_{12} + l_1 l_2 c_1 s_{12} + l_2^2 s_{12} c_{12} = 0$$

$$-l_1 l_2 s_1 c_{12} + l_1 l_2 c_1 s_{12} = 0$$

$$c_1 s_{12} - s_1 c_{12} = 0$$

$$c_1 (s_1 c_2 + c_1 s_2) - s_1 (c_1 c_2 - s_1 s_2) = 0$$

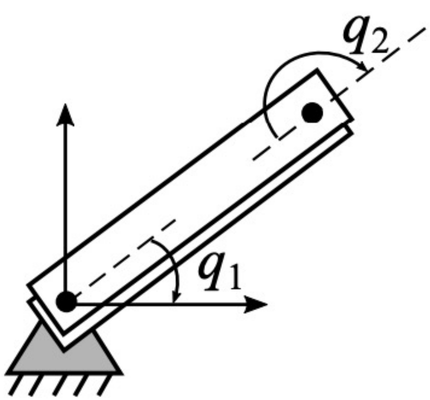
$$s_1 c_1 c_2 + c_1^2 s_2 - s_1 c_1 c_2 + s_1^2 s_2 = 0$$

$$s_2 = 0$$

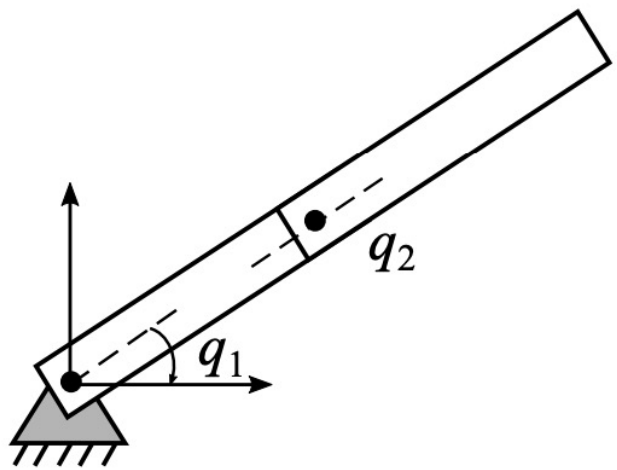
$$q_2 = 0 \text{ or } q_2 = \pi$$

When is Jacobian non-invertible?

$$q_2 = 0 \text{ or } q_2 = \pi$$



(a) $q_2 = \pi$



(b) $q_2 = 0$

Kinematic Singularities

- Let $q_2 = 0$:

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$

Kinematic Singularities

- Let $q_2 = 0$:

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 s_1 - l_2 s_1 & -l_2 s_1 \\ l_1 c_1 + l_2 c_1 & l_2 c_1 \end{bmatrix}$$

Kinematic Singularities

- Let $q_2 = 0$:

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 s_1 - l_2 s_1 & -l_2 s_1 \\ l_1 c_1 + l_2 c_1 & l_2 c_1 \end{bmatrix}$$

$$= \begin{bmatrix} -(l_1 + l_2) s_1 & -l_2 s_1 \\ (l_1 + l_2) c_1 & l_2 c_1 \end{bmatrix}$$

Kinematic Singularities

- Let $q_2 = 0$:

$$J = \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 s_1 - l_2 s_1 & -l_2 s_1 \\ l_1 c_1 + l_2 c_1 & l_2 c_1 \end{bmatrix}$$

$$= \begin{bmatrix} -(l_1 + l_2) s_1 & -l_2 s_1 \\ (l_1 + l_2) c_1 & l_2 c_1 \end{bmatrix}$$

Kinematic Singularities

- Mobility of the structure is reduced
- Small velocities of the end-effector may result in large joint velocities

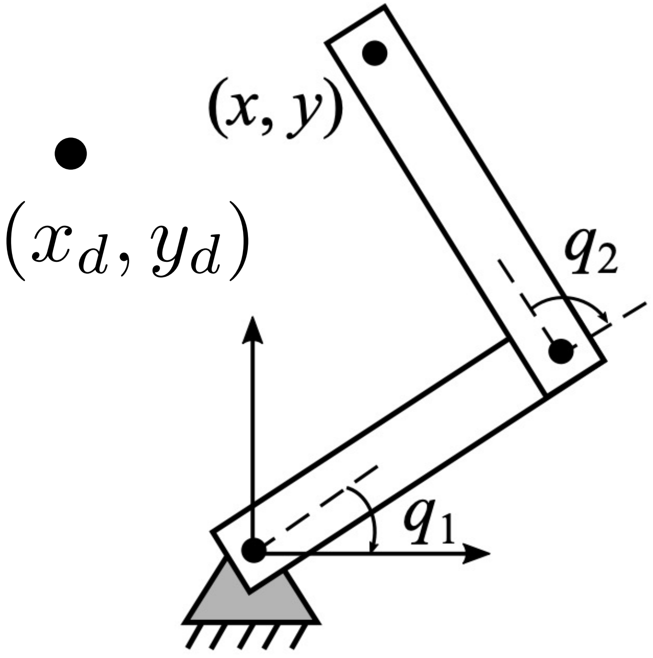
Example: 2-R Planar Manipulator

- We can compute the angular velocities of the joints, given the desired motion of the end-effector

- Can we use this for inverse kinematics?

Objective

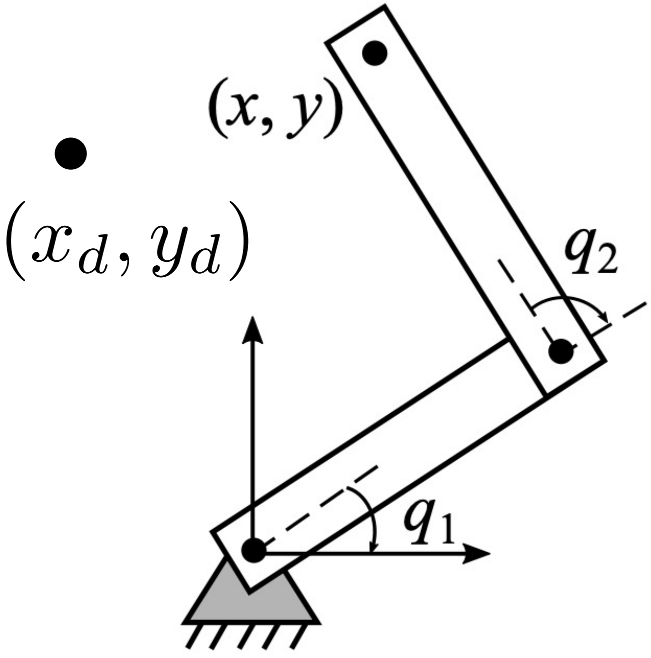
$$F = \frac{1}{2}(p_d - p)^T(p_d - p)$$



How can we solve this optimization problem?

Objective

$$F = \frac{1}{2}(p_d - p)^T(p_d - p)$$

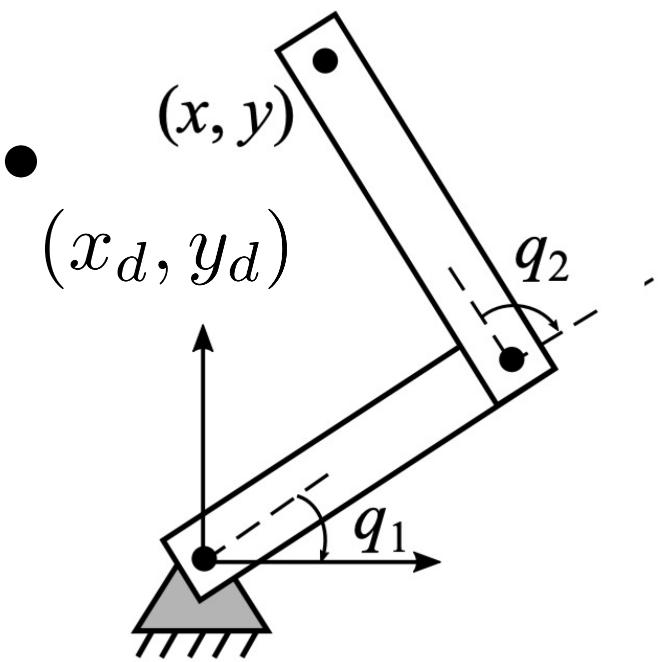


How can we solve this optimization problem?

Gradient Descent!

Inverse Kinematics with Gradient Descent

$$F = \frac{1}{2}(p_d - p)^T(p_d - p)$$

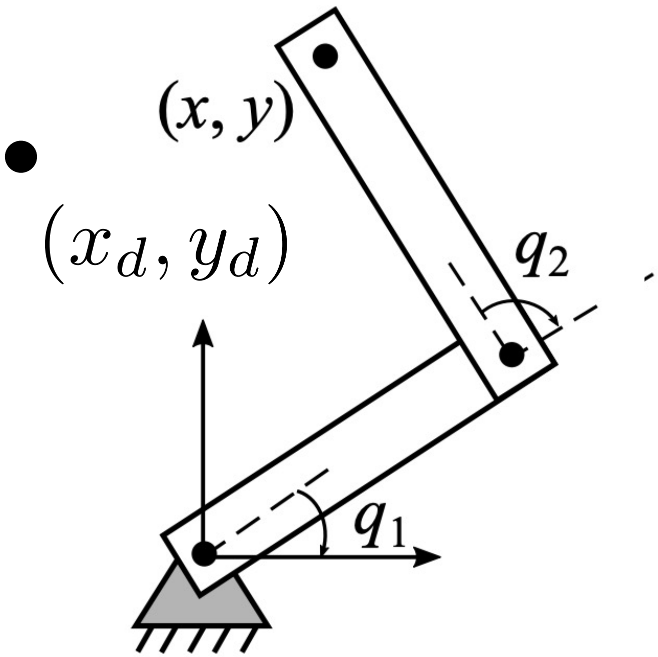


$$q_{t+1} = q_t - \alpha[\nabla_q F]^T$$

$$\nabla_q F = \begin{bmatrix} \frac{\partial F}{q_1} & \cdots & \frac{\partial F}{q_n} \end{bmatrix}$$

Inverse Kinematics with Gradient Descent

$$F = \frac{1}{2}(p_d - p)^T(p_d - p)$$



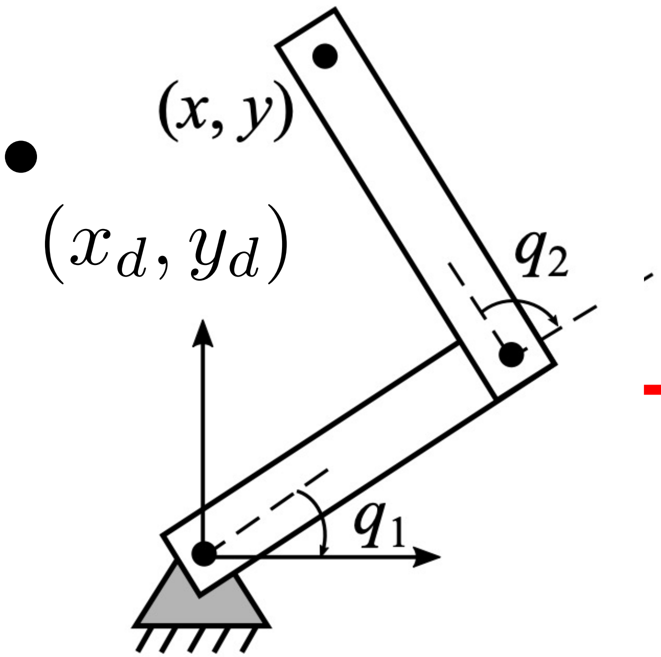
$$q_{t+1} = q_t - \alpha[\nabla_q F]^T$$

$$\nabla_q F = \begin{bmatrix} \frac{\partial F}{q_1} & \cdots & \frac{\partial F}{q_n} \end{bmatrix}$$

$$\nabla_q F = -(p_d - p)^T \nabla_q \phi(q)$$

Inverse Kinematics with Gradient Descent

$$F = \frac{1}{2}(p_d - p)^T(p_d - p)$$



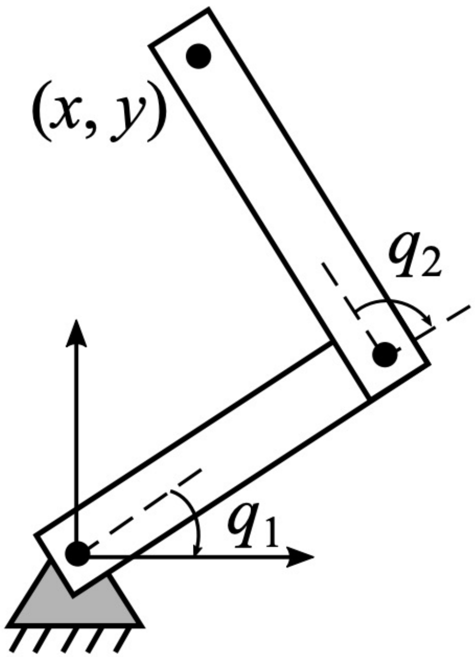
$$q_{t+1} = q_t - \alpha[\nabla_q F]^T$$

$$\nabla_q F = \begin{bmatrix} \frac{\partial F}{q_1} & \cdots & \frac{\partial F}{q_n} \end{bmatrix}$$

$$\nabla_q F = -(p_d - p)^T J(q)$$

Example: 2-R Planar Manipulator

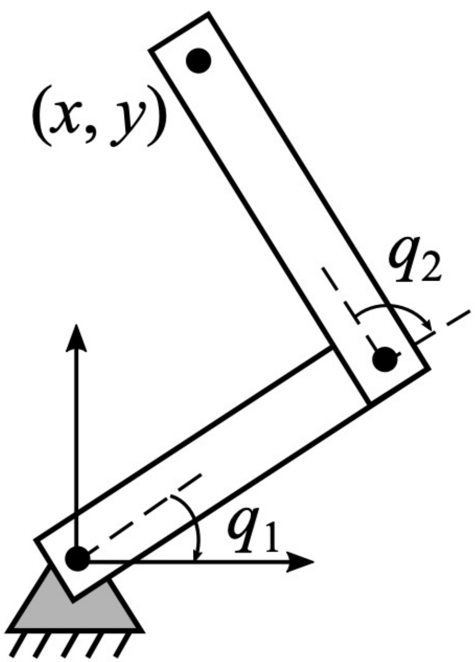
$$p_d = \begin{bmatrix} x_d \\ y_d \end{bmatrix}, p = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12}) \end{bmatrix}$$



Example: 2-R Planar Manipulator

$$p_d = \begin{bmatrix} x_d \\ y_d \end{bmatrix}, p = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12}) \end{bmatrix}$$

$$F = \frac{1}{2} \begin{bmatrix} x_d - l_1 c_1 - l_2 c_{12} \\ y_d - l_1 s_1 - l_2 s_{12} \end{bmatrix}^T \begin{bmatrix} x_d - l_1 c_1 - l_2 c_{12} \\ y_d - l_1 s_1 - l_2 s_{12} \end{bmatrix}$$

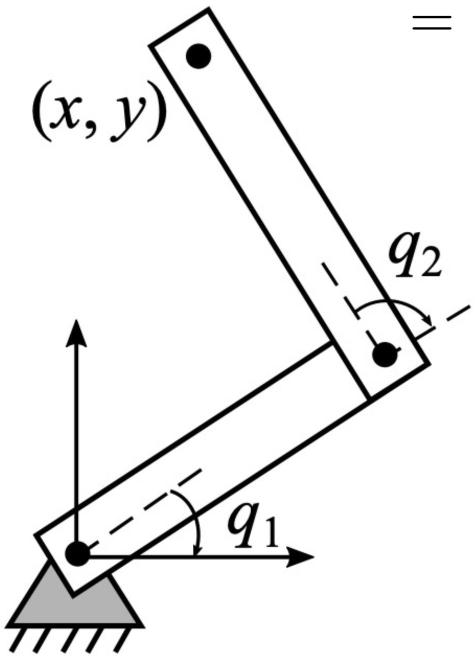


Example: 2-R Planar Manipulator

$$p_d = \begin{bmatrix} x_d \\ y_d \end{bmatrix}, p = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12}) \end{bmatrix}$$

$$F = \frac{1}{2} \begin{bmatrix} x_d - l_1 c_1 - l_2 c_{12} \\ y_d - l_1 s_1 - l_2 s_{12} \end{bmatrix}^T \begin{bmatrix} x_d - l_1 c_1 - l_2 c_{12} \\ y_d - l_1 s_1 - l_2 s_{12} \end{bmatrix}$$

$$= \frac{1}{2} [(x_d - l_1 c_1 - l_2 c_{12})^2 + (y_d - l_1 s_1 - l_2 s_{12})^2]$$



Example: 2-R Planar Manipulator

$$p_d = \begin{bmatrix} x_d \\ y_d \end{bmatrix}, p = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12}) \end{bmatrix}$$

$$\begin{aligned} F &= \frac{1}{2} \begin{bmatrix} x_d - l_1 c_1 - l_2 c_{12} \\ y_d - l_1 s_1 - l_2 s_{12} \end{bmatrix}^T \begin{bmatrix} x_d - l_1 c_1 - l_2 c_{12} \\ y_d - l_1 s_1 - l_2 s_{12} \end{bmatrix} \\ &= \frac{1}{2} [(x_d - l_1 c_1 - l_2 c_{12})^2 + (y_d - l_1 s_1 - l_2 s_{12})^2] \\ \frac{\partial F}{\partial q_1} &= (x_d - l_1 c_1 - l_2 c_{12})(l_1 s_1 + l_2 s_{12}) \\ &+ (y_d - l_1 s_1 - l_2 s_{12})(-l_1 c_1 - l_2 c_{12}) \\ &= -(p_d - p)^T \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \end{bmatrix} \end{aligned}$$

Example: 2-R Planar Manipulator

$$p_d = \begin{bmatrix} x_d \\ y_d \end{bmatrix}, p = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12}) \end{bmatrix}$$

$$\begin{aligned} F &= \frac{1}{2} \begin{bmatrix} x_d - l_1 c_1 - l_2 c_{12} \\ y_d - l_1 s_1 - l_2 s_{12} \end{bmatrix}^T \begin{bmatrix} x_d - l_1 c_1 - l_2 c_{12} \\ y_d - l_1 s_1 - l_2 s_{12} \end{bmatrix} \\ &= \frac{1}{2} [(x_d - l_1 c_1 - l_2 c_{12})^2 + (y_d - l_1 s_1 - l_2 s_{12})^2] \end{aligned}$$

$$\begin{aligned} \frac{\partial F}{\partial q_1} &= (x_d - l_1 c_1 - l_2 c_{12})(l_1 s_1 + l_2 s_{12}) \\ &+ (y_d - l_1 s_1 - l_2 s_{12})(-l_1 c_1 - l_2 c_{12}) \\ &= -(p_d - p)^T \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \end{bmatrix} \end{aligned}$$

$$\frac{\partial F}{\partial q_2} = -(p_d - p)^T \begin{bmatrix} -l_2 s_{12} \\ l_2 c_{12} \end{bmatrix}$$

Example: 2-R Planar Manipulator

$$p_d = \begin{bmatrix} x_d \\ y_d \end{bmatrix}, p = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12}) \end{bmatrix}$$

$$F = \frac{1}{2} \begin{bmatrix} x_d - l_1 c_1 - l_2 c_{12} \\ y_d - l_1 s_1 - l_2 s_{12} \end{bmatrix}^T \begin{bmatrix} x_d - l_1 c_1 - l_2 c_{12} \\ y_d - l_1 s_1 - l_2 s_{12} \end{bmatrix}$$

$$= \frac{1}{2} [(x_d - l_1 c_1 - l_2 c_{12})^2 + (y_d - l_1 s_1 - l_2 s_{12})^2]$$

$$\frac{\partial F}{\partial q_1} = (x_d - l_1 c_1 - l_2 c_{12})(l_1 s_1 + l_2 s_{12})$$

$$+ (y_d - l_1 s_1 - l_2 s_{12})(-l_1 c_1 - l_2 c_{12})$$

$$= -(p_d - p)^T \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \end{bmatrix}$$

$$\frac{\partial F}{\partial q_2} = -(p_d - p)^T \begin{bmatrix} -l_2 s_{12} \\ l_2 c_{12} \end{bmatrix}$$

$$\nabla_q F = [\frac{\partial F}{\partial q_1}, \frac{\partial F}{\partial q_2}] = -(p_d - p)^T \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

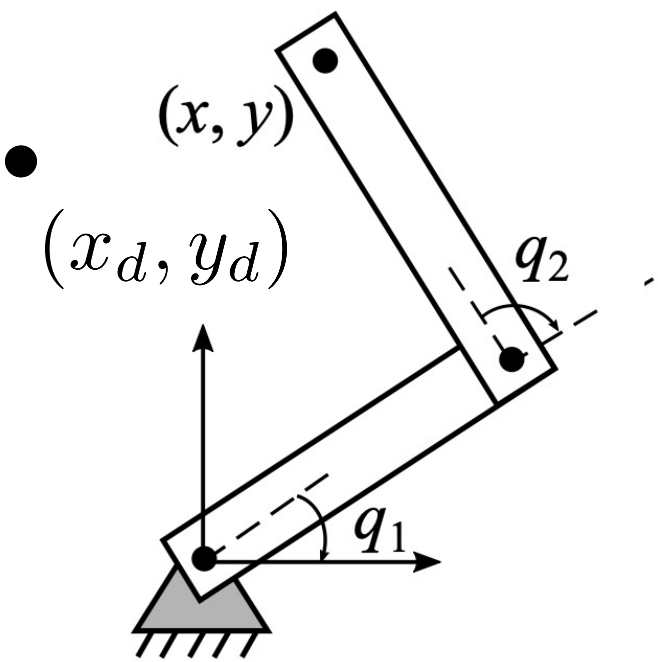
$$= -(p_d - p)^T J(q)$$

Example: 2-R Planar Manipulator

$$
\begin{aligned}
J &= \nabla_q \phi \\
&= \begin{bmatrix} \frac{\partial \phi_1}{\partial q_1} & \frac{\partial \phi_1}{\partial q_2} \\ \frac{\partial \phi_2}{\partial q_1} & \frac{\partial \phi_2}{\partial q_2} \end{bmatrix} \\
&= \begin{bmatrix} -l_1 sin(q_1) - l_2 sin(q_1 + q_2) & -l_2 sin(q_1 + q_2) \\ l_1 cos(q_1) + l_2 cos(q_1 + q_2) & l_2 cos(q_1 + q_2) \end{bmatrix}
\end{aligned}
$$

Inverse Kinematics with Gradient Descent

$$F = \frac{1}{2}(p_d - p)^T(p_d - p)$$



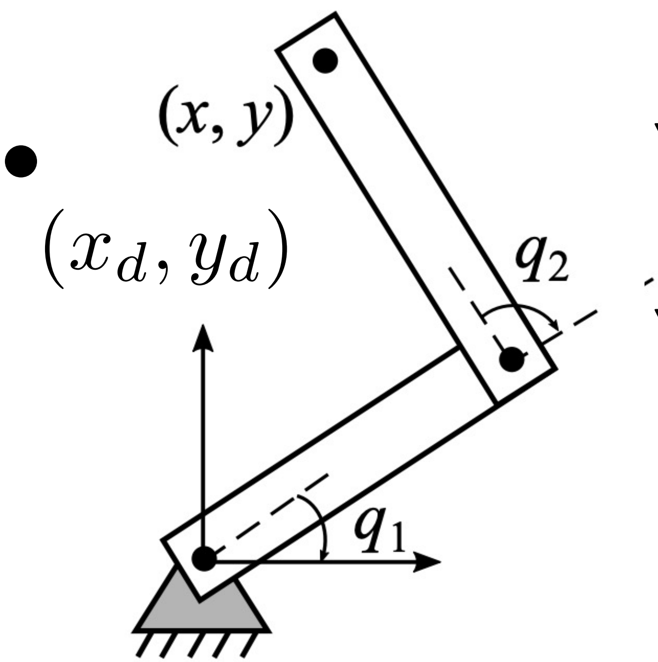
$$q_{t+1} = q_t - \alpha[\nabla_q F]^T$$

$$\nabla_q F = \begin{bmatrix} \frac{\partial F}{q_1} & \cdots & \frac{\partial F}{q_n} \end{bmatrix}$$

$$\nabla_q F = -(p_d - p)^T \nabla_q \phi(q)$$

Inverse Kinematics with Gradient Descent

$$F = \frac{1}{2}(p_d - p)^T(p_d - p)$$



$$q_{t+1} = q_t - \alpha[\nabla_q F]^T$$

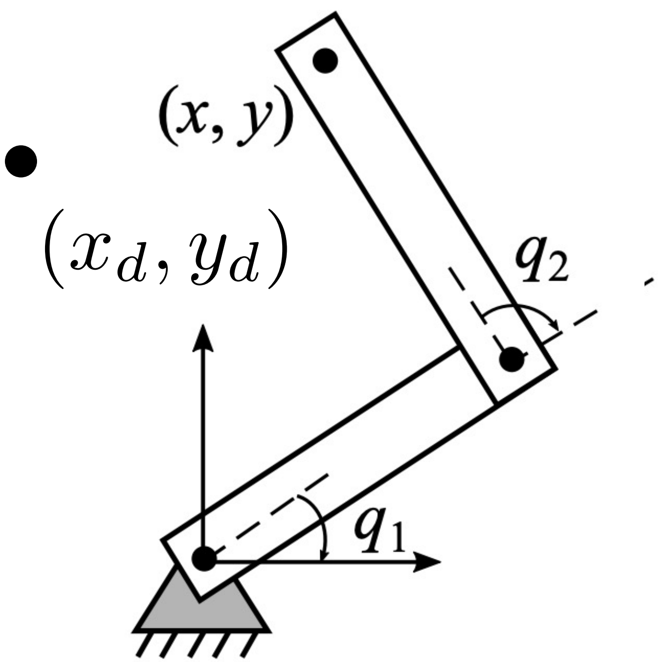
$$\nabla_q F = \begin{bmatrix} \frac{\partial F}{q_1} & \cdots & \frac{\partial F}{q_n} \end{bmatrix}$$

$$\nabla_q F = -(p_d - p)^T \nabla_q \phi(q)$$

$$\begin{aligned} q_{t+1} &= q_t + \alpha \nabla_q \phi(q)^T (p_d - p) \\ &= q_t + \alpha J(q)^T (p_d - p) \end{aligned}$$

Inverse Kinematics with Gradient Descent

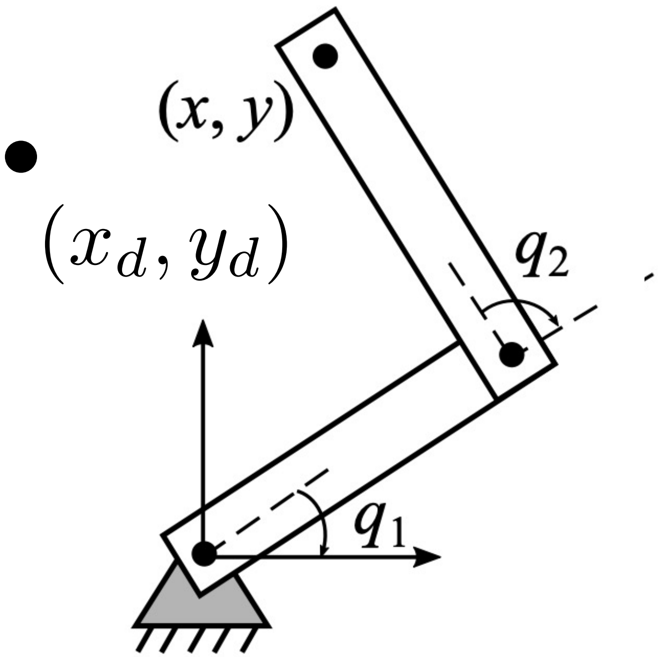
$$F = \frac{1}{2}(p_d - p)^T(p_d - p)$$



$$\Delta q = \alpha J^T \Delta p$$

Inverse Kinematics with Gradient Descent

$$F = \frac{1}{2}(p_d - p)^T(p_d - p)$$

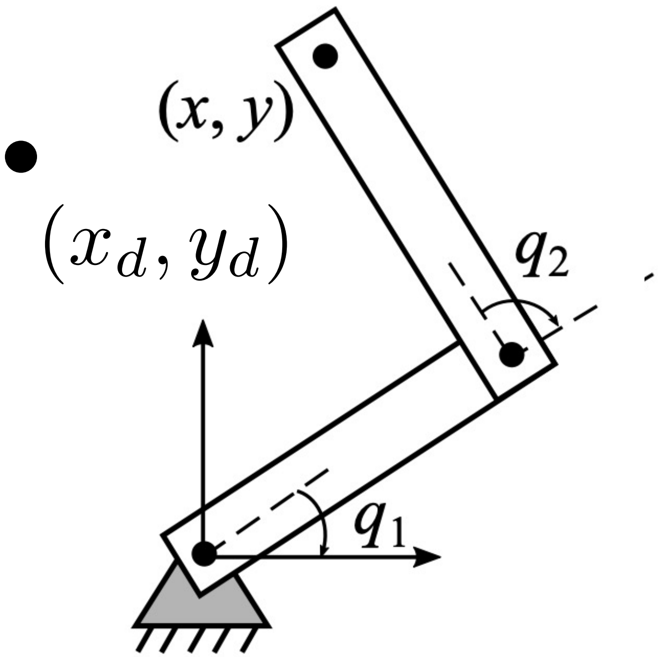


$$\Delta q = \alpha J^T \Delta p$$

Why do we need multiple iterations?

Inverse Kinematics as Constrained Optimization Problem

$$F = \frac{1}{2}\Delta q^T \Delta q$$

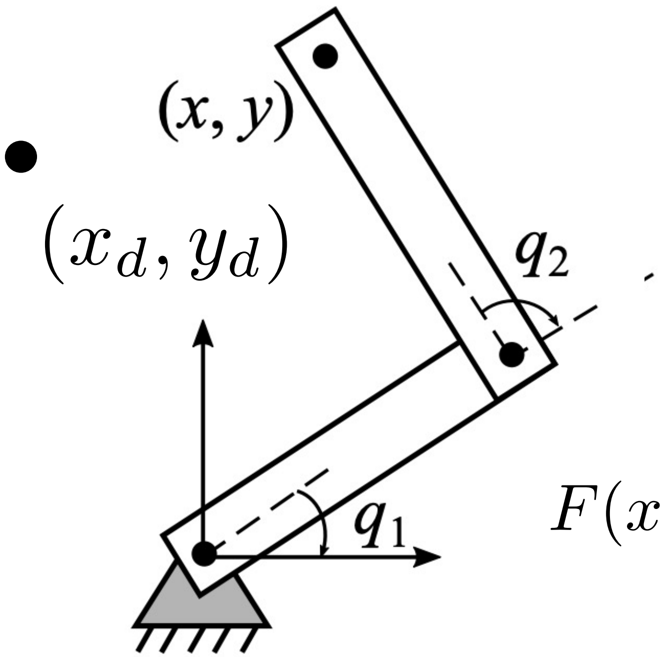


s.t : $$\Delta p = J\Delta q$$

How to solve this optimization problem?

Inverse Kinematics as Constrained Optimization Problem

$$F = \frac{1}{2}\Delta q^T \Delta q$$



s.t : $\Delta p = J \Delta q$

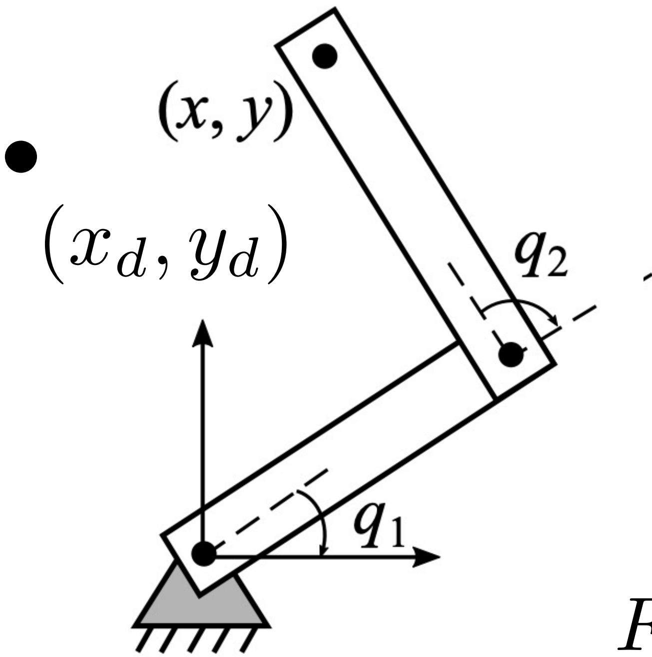
How to solve this optimization problem?

$$F(x_1, ..., x_n, \lambda) = f(x_1, ..., x_n) + \lambda g(x_1, ..., x_n)$$

where $g(x_1, ..., x_n) = 0$

Inverse Kinematics as Constrained Optimization Problem

$$F = \frac{1}{2}\Delta q^T \Delta q$$



s.t : $\Delta p = J \Delta q$

How to solve this optimization problem?

$$F(\Delta q, \lambda) = \frac{1}{2}\Delta q^T \Delta q + \lambda(\Delta p - J\Delta q)$$

Inverse Kinematics as Constrained Optimization Problem

$$F(\Delta q, \lambda) = \frac{1}{2}\Delta q^T \Delta q + \lambda(\Delta p - J\Delta q)$$

Inverse Kinematics as Constrained Optimization Problem

$$F(\Delta q, \lambda) = \frac{1}{2}\Delta q^T \Delta q + \lambda(\Delta p - J\Delta q)$$

$$\frac{\partial F}{\partial \Delta q} = 0 \rightarrow$$

$$\Delta q - \lambda J^T = 0 \rightarrow$$

$$\Delta q = J^T \lambda$$

Inverse Kinematics as Constrained Optimization Problem

$$F(\Delta q, \lambda) = \frac{1}{2}\Delta q^T \Delta q + \lambda(\Delta p - J\Delta q)$$

$$\frac{\partial F}{\partial \Delta q} = 0 \rightarrow$$

$$\Delta q - \lambda J^T = 0 \rightarrow$$

$$\Delta q = J^T \lambda$$

$$\frac{\partial F}{\partial \lambda} = 0 \rightarrow$$

$$\Delta p = J\Delta q$$

Inverse Kinematics as Constrained Optimization Problem

$$\Delta q = J^T \lambda$$

$$\Delta p = J \Delta q$$

Inverse Kinematics as Constrained Optimization Problem

$$\Delta q = J^T \lambda$$
$$\Delta p = J \Delta q$$

$$\longrightarrow$$

$$J \Delta q = J J^T \lambda$$
$$\Delta p = J \Delta q$$

Inverse Kinematics as Constrained Optimization Problem

$$\Delta q = J^T \lambda \qquad \longrightarrow \qquad J\Delta q = JJ^T \lambda \qquad \longrightarrow \qquad \Delta p = JJ^T \lambda$$

$$\Delta p = J\Delta q \qquad \qquad \qquad \Delta p = J\Delta q \qquad \qquad \qquad \Delta p = J\Delta q$$

Inverse Kinematics as Constrained Optimization Problem

$$\frac{\Delta q = J^T \lambda}{\Delta p = J \Delta q} \longrightarrow \begin{matrix} J \Delta q = J J^T \lambda \\ \Delta p = J \Delta q \end{matrix} \longrightarrow \frac{\Delta p = J J^T \lambda}{\Delta p = J \Delta q}$$

$$\Delta q = \underbrace{J^T (J J^T)^{-1}}_{J^{\#}} \Delta p$$

Jacobian Pseudoinverse

Inverse Kinematics as Constrained Optimization Problem

$$\Delta q = \underbrace{J^T(JJ^T)^{-1}}_{J^{\#}}\Delta p$$

Jacobian Pseudoinverse

Gradient descent:

$$q_{t+1} = q_t + \alpha J^{\#}\Delta p$$

General Formulation of IK Problem

$$\min\ J = ||p - p_d||$$

Constraints:

- joint limits
- kinematics
- collision avoidance