Dynamics

CSCI 545 Introduction to Robotics
Instructor: Stefanos Nikolaidis

Resources: CMU 16-811: Math Fundamentals for Robotics
by Prof. Michael Erdmann

Introduction

- Newton's equations are typically specified for point masses

- But what about generalized coordinates, i.e., coordinates beyond Cartesian?

- Example: joint configurations $q_1, \ldots, q_n$

Generalized Forces

- Joint configurations $q_1, \ldots, q_n$

- Generalized forces $Q_1, \ldots, Q_n$

- A generalized force measures the amount of work done in the q_i direction due to an infinitesimal displacement of the system:

$$\text{Work Done} = \sum_{i=1}^{r} Q_i \delta q_i$$

Example

- Suppose we have *N* particles (point masses in 3D space)
- If the particles don’t have any constraints, we may describe the state of the system with ?? coordinates

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- If the particles don't have any constraints, we may describe the state of the system with 3N coordinates

- We let the coordinates be the positions $r_1, \ldots, r_n$ of the particles

- The forces applied are the Newtonian forces acting on the particles $F_1, \ldots, F_n$

Example

- Suppose we have *N* particles (point masses in 3D space)
- If the particles don't have any constraints, we may describe the state of the system with 3N coordinates
- We let the coordinates be the positions $r_1, \ldots, r_n$ of the particles
- The forces applied are the Newtonian forces acting on the particles
- What if there are k (possibly time-varying) holonomic constraints?

Example

- For example, we let the rigid connection of the particles is expressed by the constraint: $|r_i - r_j|^2 - d_{ij}^2 = 0$

- We can then express the old coordinates of the particles with respect to 3N-k new (generalized) coordinates:

$$r_1 = r_1(q_1, ..., q_{3N-K}, t)$$

$$\vdots$$

$$r_N = r_N(q_1, ..., q_{3N-K}, t)$$

Example

- Let F_i the force acting on particle *i*
- The total work done by the forces F_i is

$$W = \sum_{i=1}^{N} F_i \delta r_i$$

Example

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Example

$$\begin{aligned} W &= \sum_{i=1}^{N} F_i \delta r_i \\ &= \sum_{i=1}^{N} F_i \sum_{j=1}^{3N-K} \frac{\delta r_i}{\delta q_j} \delta q_j \end{aligned}$$

Example

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\begin{aligned}
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&= \sum_{i=1}^{N} F_i \sum_{j=1}^{3N-K} \frac{\delta r_i}{\delta q_j} \delta q_j \\
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Example

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Example

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&= \sum_{j=1}^{3N-K} \underline{Q_j} \delta q_j
\end{aligned}
$$

Generalized Force

Lagrange Equations

We can then derive Lagrange's equations from Newton's equations

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = Q_j, j = 1, ..., r$$

Lagrange Equations

We can then derive Lagrange's equations from Newton's equations

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = Q_j, j = 1, ..., r$$

If the forces are derived from a potential *V*,

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = Q_j, j = 1, ..., r$$

Where L = T – V the Lagrangian of the system

Example

- For an 1D particle, the Kinetic energy is:

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$$\frac{dT}{d\dot{q}} = m\dot{x}$$

$$\frac{d}{dt}\frac{dT}{d\dot{q}} = m\ddot{x}$$

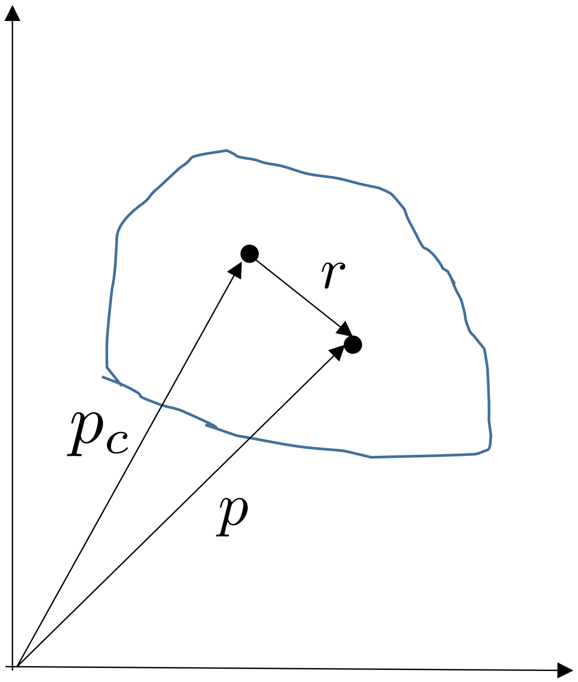
Robot Manipulator

- To apply Lagrange's equations to a robotic manipulator, we calculate the kinetic and potential energy of the robot's link as a function of the robot's generalized coordinates (joint angles and velocities)

- In turn, this requires modeling the mass distribution of the links

Rigid Body Motion

$$V \in \mathbb{R}^3, \rho(r), r \in V$$



$$m = \int \rho(r) dV$$

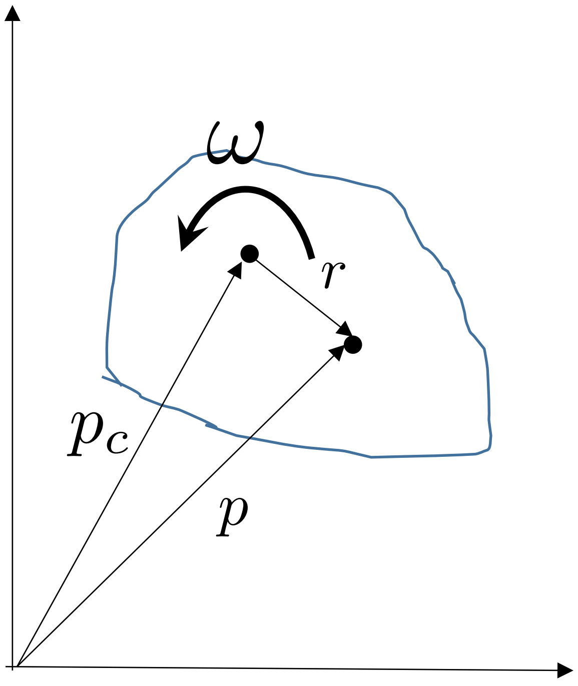
- For the center of mass:

$$p_c = \frac{1}{m} \int \rho(r) p dV$$

Rigid Body Motion

$$\dot{p} = \dot{p}_c + \omega \times r$$
$$= \dot{p}_c + S(w)r$$

$$S(w) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$



Kinetic Energy

$$T = \int \frac{1}{2}\rho(r)\dot{p}^T\dot{p}dV$$

Assuming homogeneous material, we have:

$$T = \rho\int \frac{1}{2}\dot{p}^T\dot{p}dV$$

Kinetic Energy

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$$T = \rho\int \frac{1}{2}\dot{p}^T\dot{p}dV$$

$$= \rho\frac{1}{2}\int(\dot{p}_c + S(w)r)^T(\dot{p}_c + S(w)r)||dV$$

Kinetic Energy

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Assuming homogeneous material, we have:

$$\begin{aligned} T &= \rho \int \frac{1}{2}\dot{p}^T\dot{p}dV \\ &= \rho\frac{1}{2}\int(\dot{p}_c + S(w)r)^T(\dot{p}_c + S(w)r)||dV \\ &= \rho\frac{1}{2}\int(||\dot{p}_c||^2 + 2\dot{p}_c^T S(w)r + (S(w)r)^T S(w)r)dV \end{aligned}$$

Kinetic Energy

$$T = \int \frac{1}{2}\rho(r)\dot{p}^T\dot{p}dV$$

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Translational Kinetic Energy

Kinetic Energy

$$T = \int \frac{1}{2}\rho(r)\dot{p}^T\dot{p}dV$$

Assuming homogeneous material, we have:

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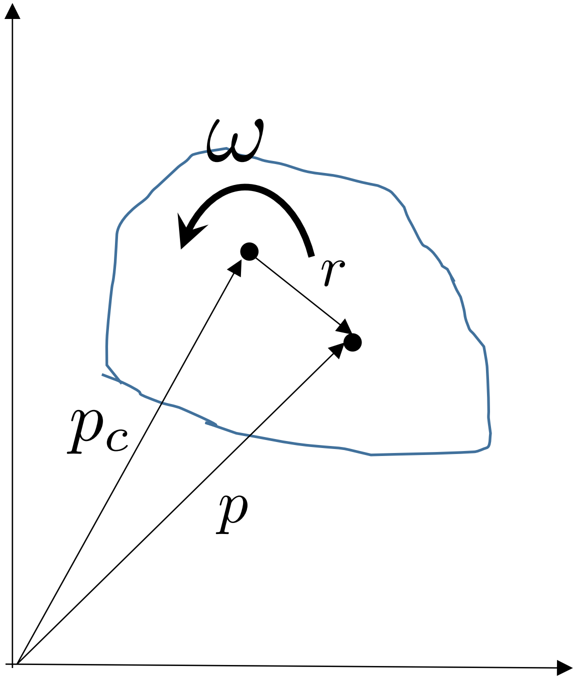
Kinetic Energy

$$\rho \frac{1}{2} \int 2\dot{p}_c^T S(w) r dV$$

$$= \rho S(w) \dot{p}_c^T \int r dV$$

$$= \rho S(w) \dot{p}_c^T \int (p - p_c) dV$$



Kinetic Energy

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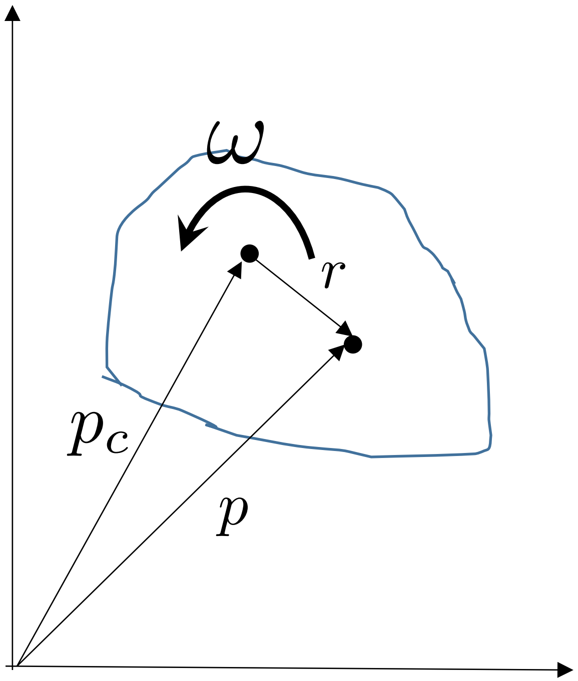

$$p_c = \frac{1}{m}\int \rho p dV$$

Kinetic Energy

$$\rho\frac{1}{2}\int 2\dot{p}_c^T S(w) r dV$$

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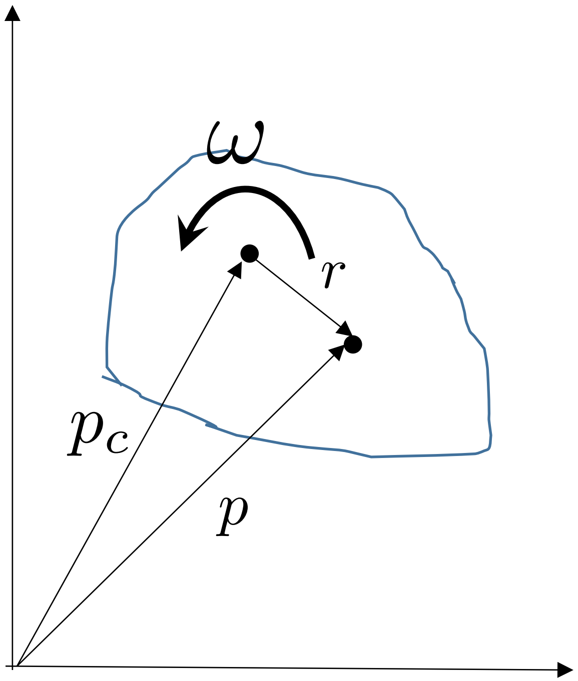
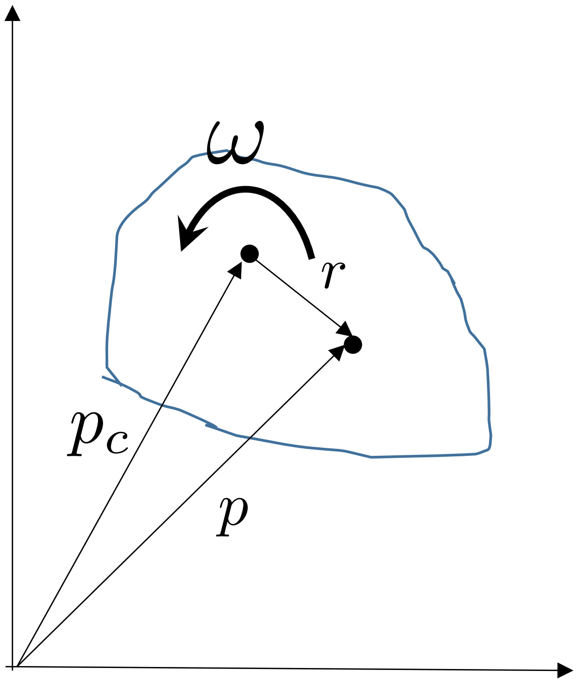


$$p_c = \frac{1}{m}\int \rho p dV$$

$$= \frac{\int \rho p dV}{\int \rho dV}$$

Kinetic Energy



$$\rho\frac{1}{2}\int 2\dot{p}_c^T S(w) r dV$$

$$= \rho S(w)\dot{p}_c^T \int r dV$$

$$= \rho S(w)\dot{p}_c^T \int (p - p_c) dV$$

$$p_c = \frac{1}{m}\int \rho p dV$$

$$= \frac{\int \rho p dV}{\int \rho dV}$$

$$\int (p_c - p)\rho dV = 0$$

Kinetic Energy

$$T = \int \frac{1}{2}\rho(r)\dot{p}^T\dot{p}dV$$

Assuming homogeneous material, we have:

$$\begin{aligned} T &= \rho\int \frac{1}{2}\dot{p}^T\dot{p}dV \\ &= \rho\frac{1}{2}\int(\dot{p}_c + S(w)r)^T(\dot{p}_c + S(w)r)||dV \\ &= \rho\frac{1}{2}\int(||\dot{p}_c||^2 + 2\dot{p}_c^T S(w)r + \underline{(S(w)r)^T S(w)r)dV} \end{aligned}$$

Kinetic Energy

$$\omega \times r = -r \times \omega$$

$$S(\omega)r = -S(r)\omega$$

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$$\rho\frac{1}{2}\int(-S(r)\omega)^T(-S(r)\omega)dV =$$

Kinetic Energy

$$\omega \times r = -r \times \omega$$

$$S(\omega)r = -S(r)\omega$$

$$\rho\frac{1}{2}\int(-S(r)\omega)^T(-S(r)\omega)dV =$$

$$\rho\frac{1}{2}\int\omega^T S(r)^T S(r)\omega dV =$$

Kinetic Energy

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$$\rho\frac{1}{2}\int \omega^T S(r)^T S(r)\omega dV =$$

$$\frac{1}{2}\omega^T \int \rho S(r)^T S(r) dV \omega =$$

Kinetic Energy

$$\omega \times r = -r \times \omega$$

$$S(\omega)r = -S(r)\omega$$

$$\rho\frac{1}{2}\int(-S(r)\omega)^T(-S(r)\omega)dV =$$

$$\rho\frac{1}{2}\int\omega^T S(r)^T S(r)\omega dV =$$

$$\frac{1}{2}\omega^T\int\rho S(r)^T S(r)dV\omega =$$

$$\frac{1}{2}\omega^T \underline{I}\omega$$

Inertia Tensor

Kinetic Energy

$$\begin{aligned} T &= \int \frac{1}{2}\rho(r)\dot{p}^T\dot{p}\,dV \\ &= \frac{1}{2}m||p_c||^2 + \frac{1}{2}\omega^T I\omega \end{aligned}$$

Translational component

Kinetic Energy

$$
\begin{aligned}
T &= \int \frac{1}{2}\rho(r)\dot{p}^T\dot{p}\,dV \\
&= \frac{1}{2}m||p_c||^2 + \underset{\text{Rotational component}}{\underline{\frac{1}{2}\omega^T I\omega}}
\end{aligned}
$$

Example: Planar Robot (Zero Gravity)

- L = T

- We can model each link with a homogeneous rectangular bar, with mass m_i and inertia tensor:

$$I_i = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

Example: Planar Robotic Arm (Zero Gravity)

- L = T

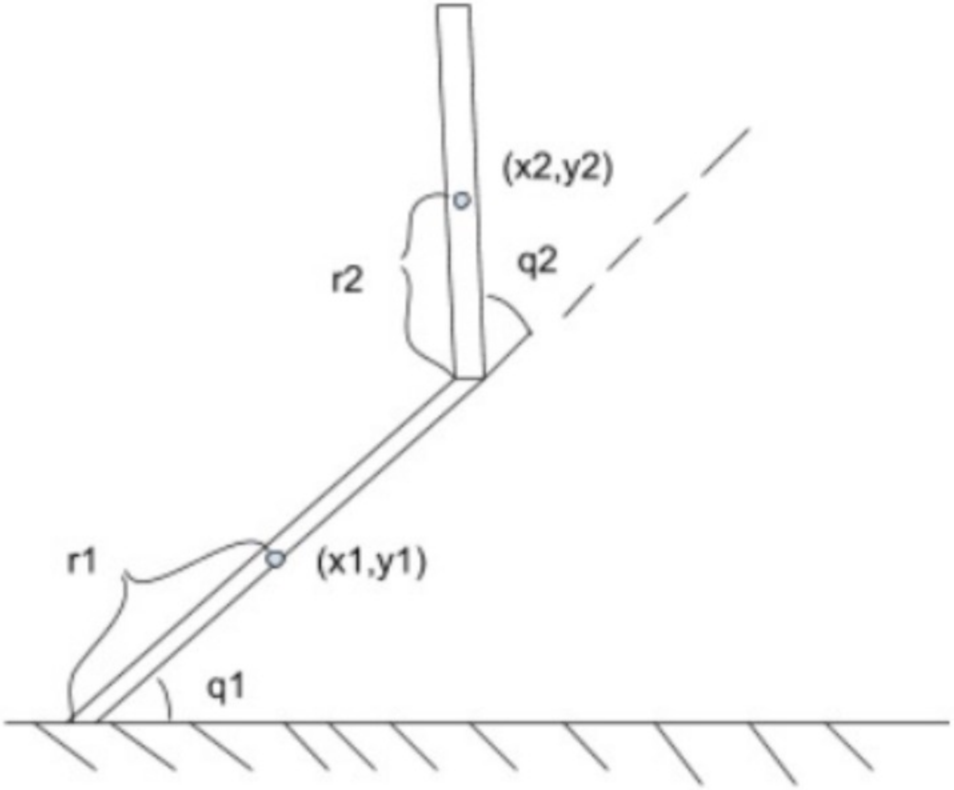
- We can model each link with a homogeneous rectangular bar, with mass m_i and inertia tensor I_i

- For each link:

$$T_i = \frac{1}{2}m_i v_i^2 + \frac{1}{2}\omega_i^T I_i \omega_i$$

Kinetic Energy for 2-R Arm

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}\omega_1^T I_1 \omega_1 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}\omega_2^T I_2 \omega_2$$



Kinetic Energy for 2-R Arm

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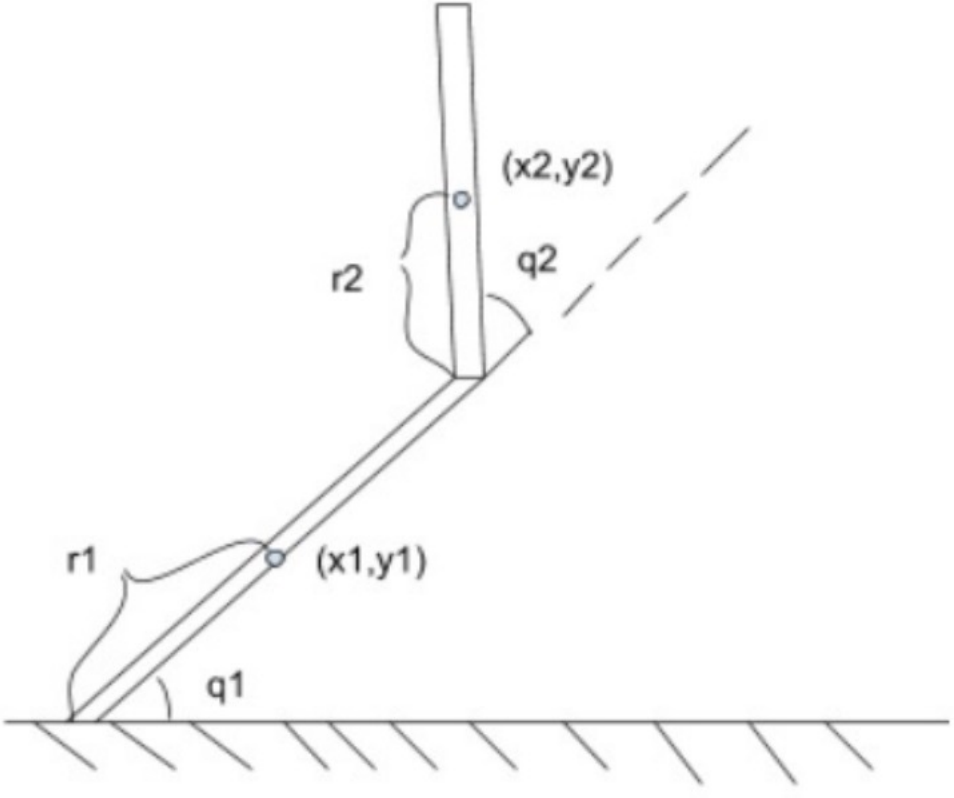
\begin{aligned}
T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}\omega_1^T I_1 \omega_1 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}\omega_2^T I_2 \omega_2 \\
&= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}[0,0,\dot{q}_1]I_1[0,0,\dot{q}_1]^T + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}[0,0,\dot{q}_1+\dot{q}_2]I_2[0,0,\dot{q}_1+\dot{q}_2]^T
\end{aligned}
$$



Kinetic Energy for 2-R Arm

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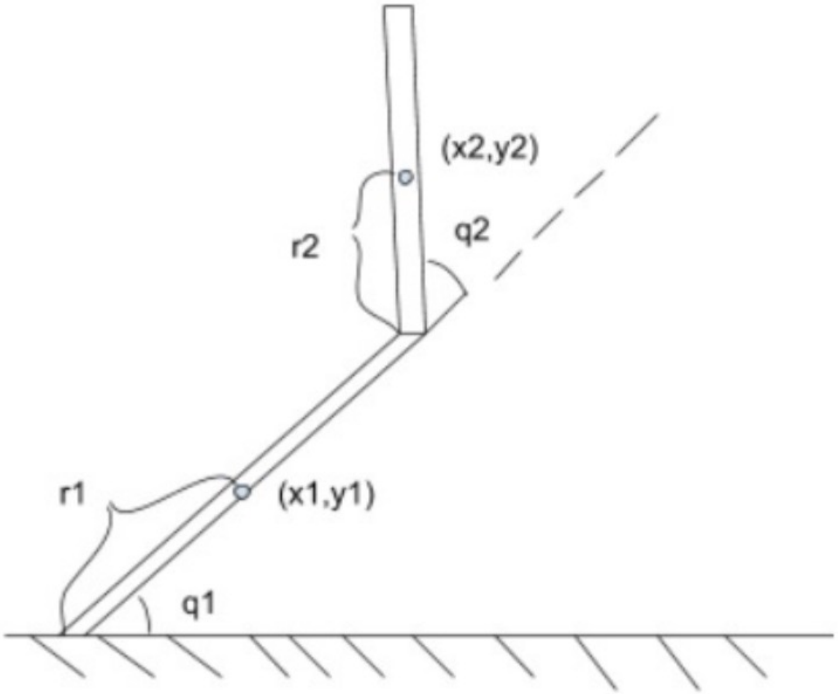
$$= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}[0, 0, \dot{q}_1] I_1 [0, 0, \dot{q}_1]^T + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}[0, 0, \dot{q}_1 + \dot{q}_2] I_2 [0, 0, \dot{q}_1 + \dot{q}_2]^T$$

$$x_1 = r_1 c_1 (= r_1 cos q_1)$$



Kinetic Energy for 2-R Arm

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\begin{aligned}
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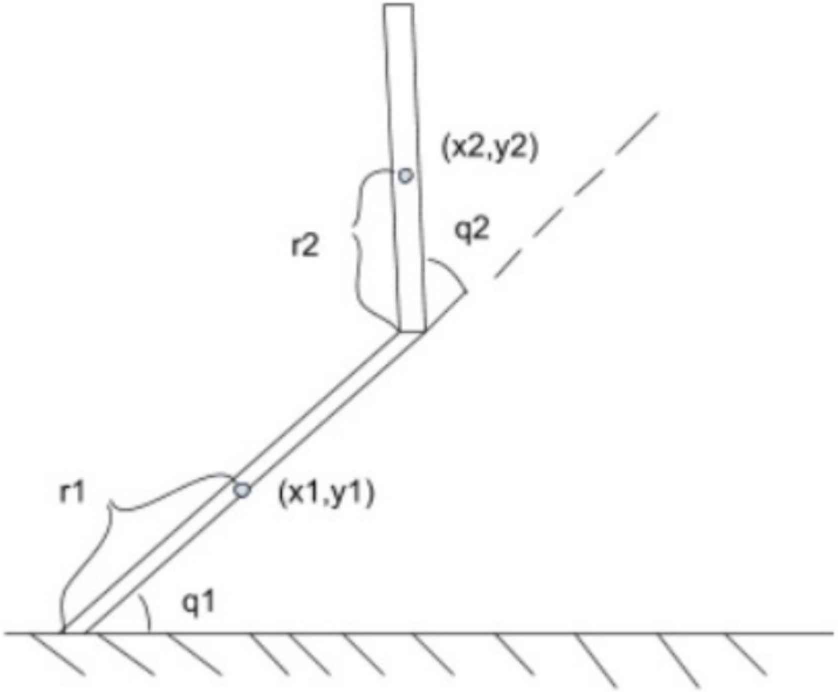
&= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}[0,0,\dot{q}_1]I_1[0,0,\dot{q}_1]^T + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}[0,0,\dot{q}_1+\dot{q}_2]I_2[0,0,\dot{q}_1+\dot{q}_2]^T
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= r_1c_1 (= r_1 cos q_1) \\
y_1 &= r_1s_1
\end{aligned}
$$



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\begin{aligned}
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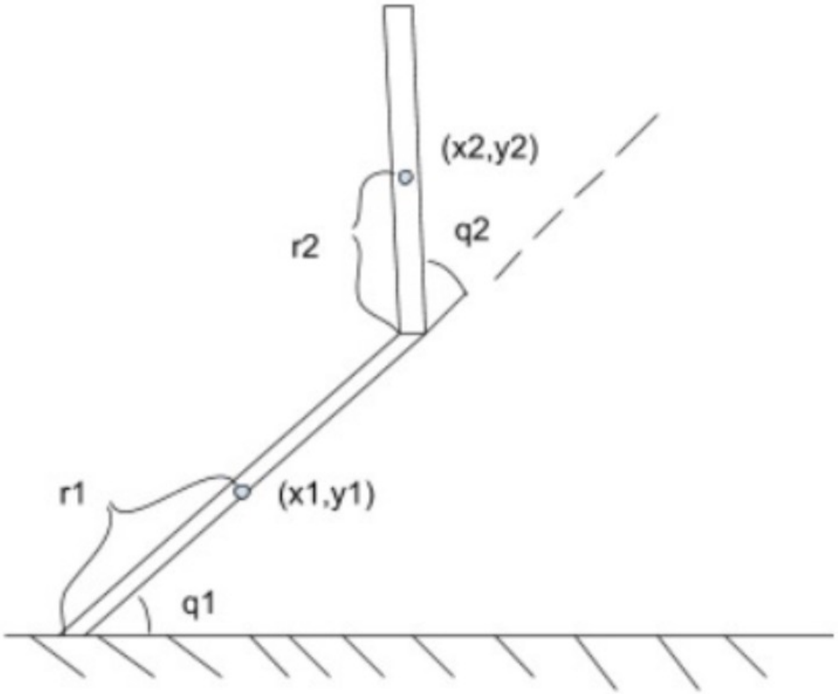
&= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}[0,0,\dot{q}_1] I_1 [0,0,\dot{q}_1]^T + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}[0,0,\dot{q}_1+\dot{q}_2] I_2 [0,0,\dot{q}_1+\dot{q}_2]^T
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= r_1 c_1 (= r_1 cos q_1) \\
y_1 &= r_1 s_1 \\
\dot{x}_1 &= -r_1 s_1 \dot{q}_1 \\
\dot{y}_1 &= r_1 c_1 \dot{q}_1
\end{aligned}
$$



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$$x_1 = r_1c_1 (= r_1 cos q_1)$$

$$y_1 = r_1 s_1$$

$$\dot{x}_1 = -r_1 s_1 \dot{q}_1$$

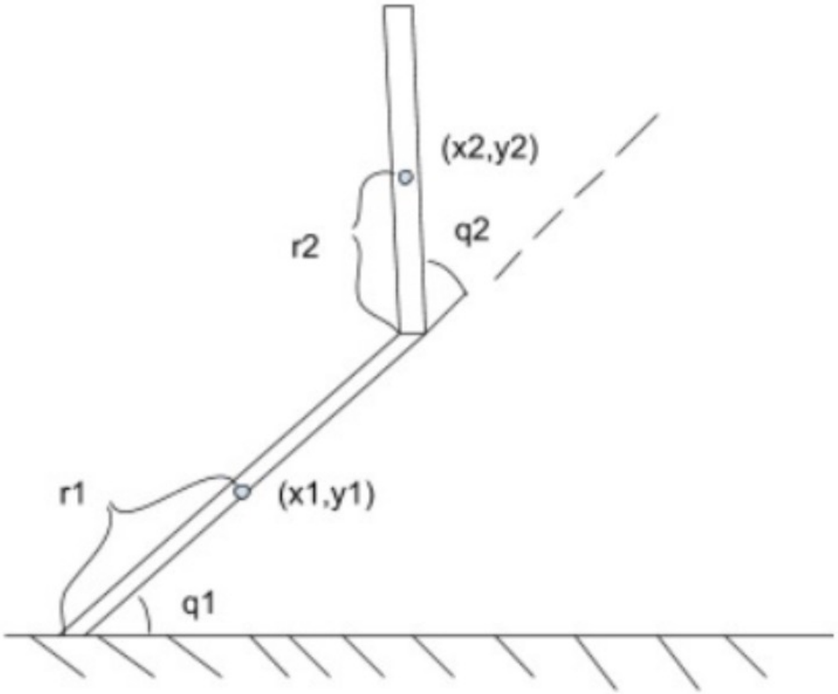
$$\dot{y}_1 = r_1 c_1 \dot{q}_1$$

$$x_2 = l_1 c_1 + r_2 c_{12}$$

$$y_2 = l_1 s_1 + r_2 s_{12}$$

$$\dot{x}_2 = -(l_1 s_1 - r_2 s_{12})\dot{q}_1 - r_2 s_{12}\dot{q}_2$$

$$\dot{y}_2 = (l_1 c_1 + r_2 c_{12})\dot{q}_1 + r_2 c_{12}\dot{q}_2$$



Kinetic Energy for 2-R Arm

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = Q_j, j = 1, ..., r$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = Q_j$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = \tau_j$$

$$B(q_1, q_2)\begin{bmatrix}\ddot{q}_1 \\ \ddot{q}_2\end{bmatrix} + C(q_1, q_2, \dot{q}_1, \dot{q}_2) = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix}$$

For General Manipulators (with Gravity)

$$L = T - U$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = Q_j$$

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For General Manipulators (with Gravity)

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

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$$\underline{B(q)}\ddot{q} + C(q, \dot{q}) + G(q) = \tau$$

positive definite inertia matrix

For General Manipulators (with Gravity)

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Centrifugal and Coriolis Forces

https://youtu.be/kCbMKSZZO9w

For General Manipulators (with Gravity)

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$$B(q)\ddot{q} + C(q, \dot{q}) + \underbrace{G(q)}_{\text{Gravitational Forces}} = \tau$$

Direct Dynamics

$$B(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau$$

We can determine $\ddot{q}$ given τ

$$\ddot{q} = B(q)^{-1}(\tau - C(q,\dot{q}) - G(q))$$

Direct Dynamics

$$B(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau$$

We can determine $\ddot{q}$ given τ

We can then compute $\dot{q}$, q using Euler integration

$$q(k+1) = q(k) + \dot{q}(k) * dt$$
$$\dot{q}(k+1) = \dot{q}(k) + \ddot{q} * dt$$

Forward Dynamics

$$B(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau$$

Why would we want to do forward dynamics?

Forward Dynamics

$$B(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau$$

We want to simulate the robot!

Inverse Dynamics

$$B(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau$$

Given. $\ddot{q}, \dot{q}, q$ compute the torques.

We can use a motion planner to specify a trajectory, then compute the torques that enable the robot to track the trajectory.

Non-linear Control

CSCI 545 Introduction to Robotics
Instructor: Stefanos Nikolaidis

Resources: CSCI 545 Lecture Notes by Prof. Stefan Scha

Dynamics Equations

$$B(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau$$

These equations are non-linear.

Non-linear Control

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