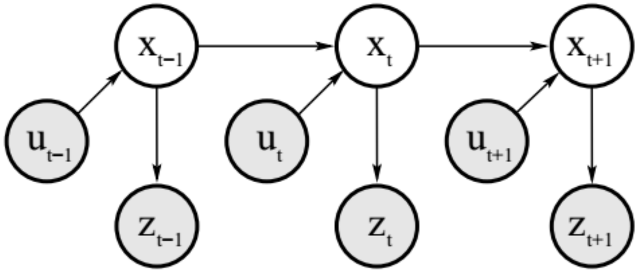
Linear Dynamical Systems

[Resources: CS 545 Introduction to Robotics taught by Prof. Stefan Schaal]

CSCI 545 Introduction to Robotics
Instructor: Stefanos Nikolaidis

Recap



$$b(x_t) = \eta P(z_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}, z_{1:t-1}, u_{1:t}) b(x_{t-1})$$

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Algorithm Bayes Filter($b(x_{t-1})$, $u_t$, $z_t$)
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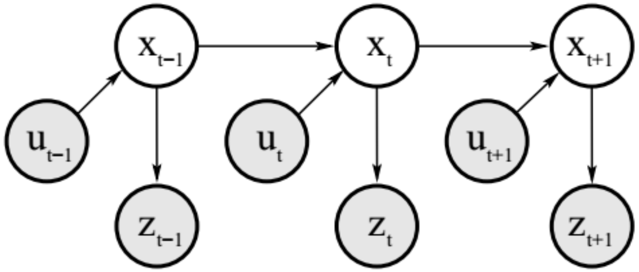
    Endfor
    Return $b(x_t)$
```

Continuous states



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Classification

Discrete State, Discrete Time	Discrete State, Continous Time
Continuous State, Discrete Time	Continuous State, Continuous Time

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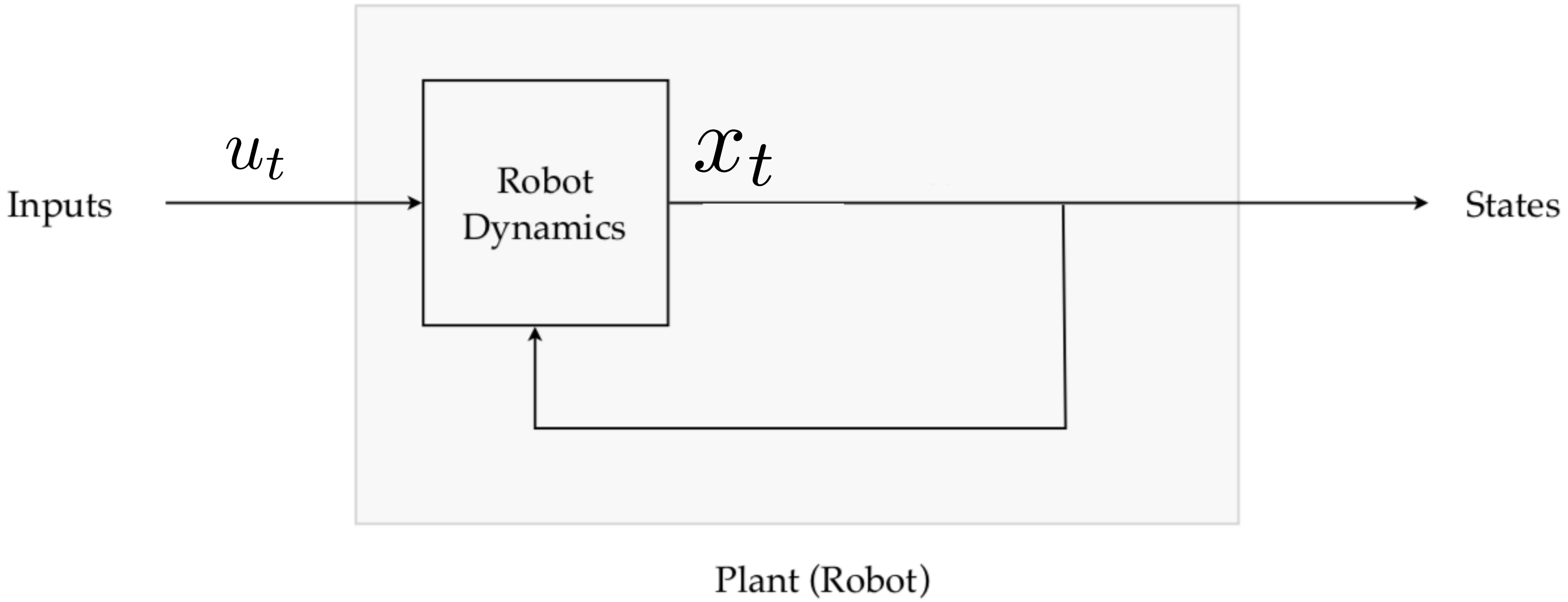
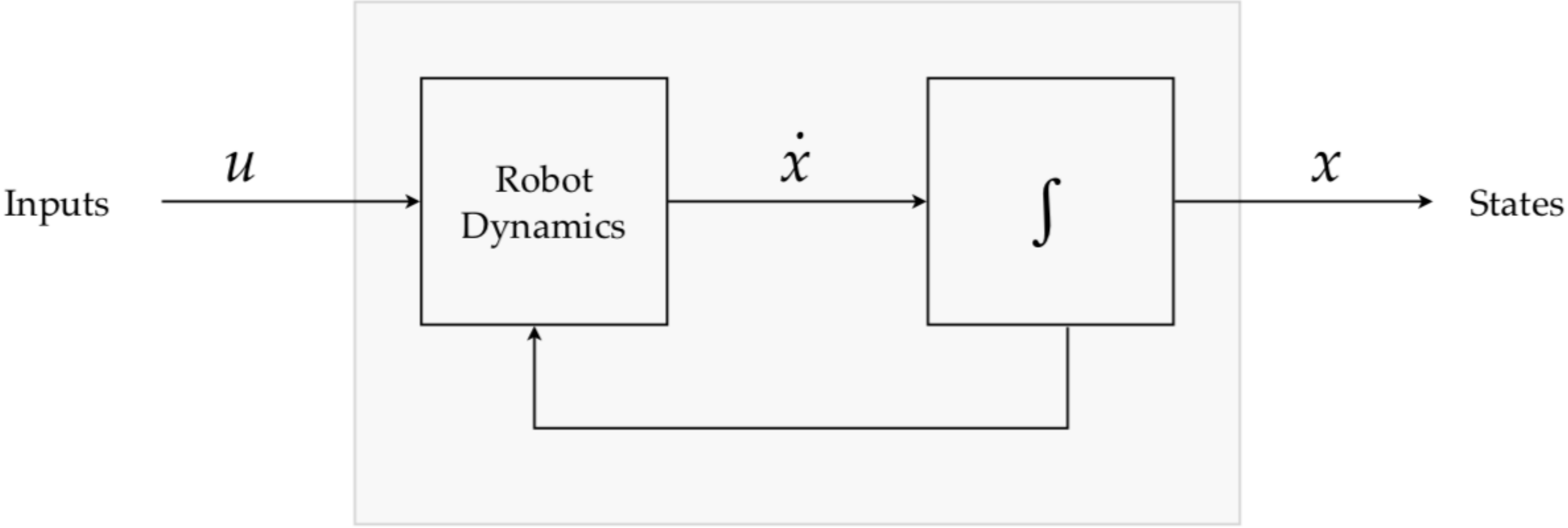
Discrete Systems



Contunuous Systems



Plant (Robot)

System Transformations

- It is always possible to transform a continuous time-system to a discrete time-system.

- Is the opposite true?

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- It is always possible to transorm a continuous time-system to a discrete time-system.

- Is the opposite true? The opposite is not true: some systems are discrete and cannot be transformed to continuous time-stsytems.

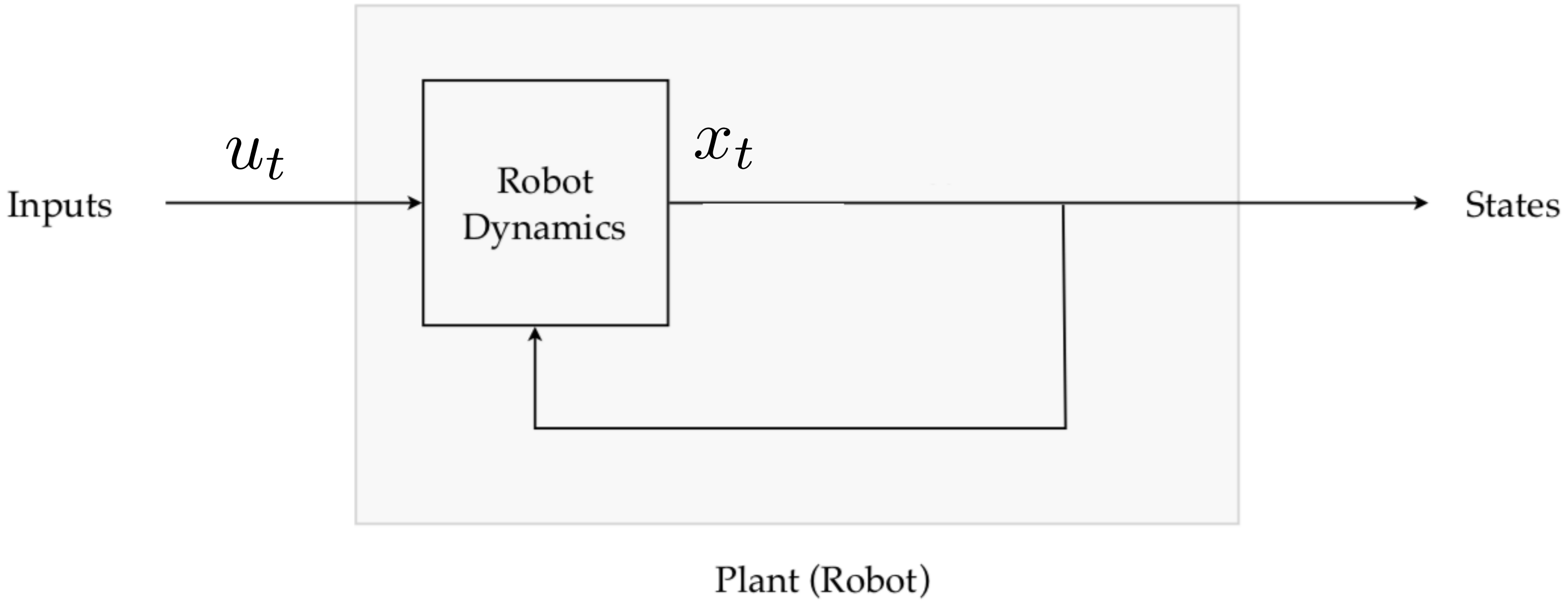
Dynamics and Output

- System Dynamics: How the system changes over time

- Output / Observation: What you observe

- Example: in a robot that has position encoders, you can measure positions but not velocities!

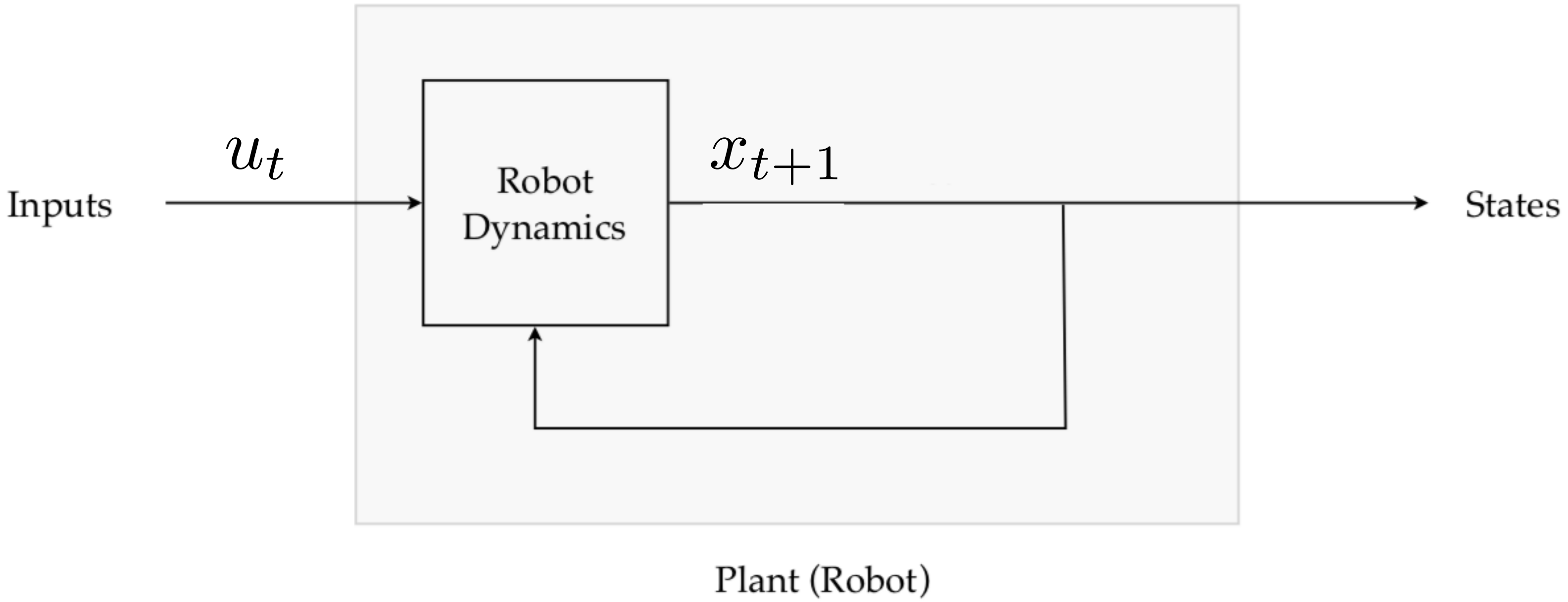
Discrete Systems



$$x_t = f(x_{t-1}, u_t, t)$$

$$z_t = g(x_t, t)$$

Stationary Discrete Systems



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Question

- How would you estimate velocities from position measurements?

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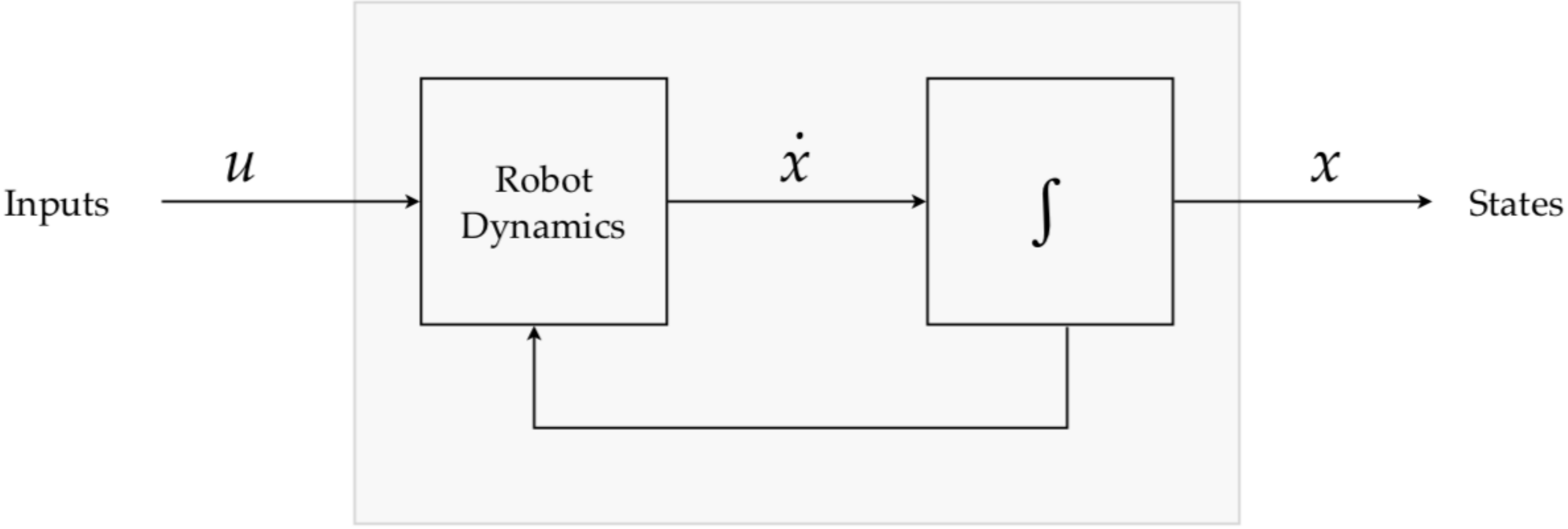
$$v_t = \frac{p_{t+1} - p_t}{\Delta_t}, \Delta_t \to 0$$

Continuous Systems



Plant (Robot)

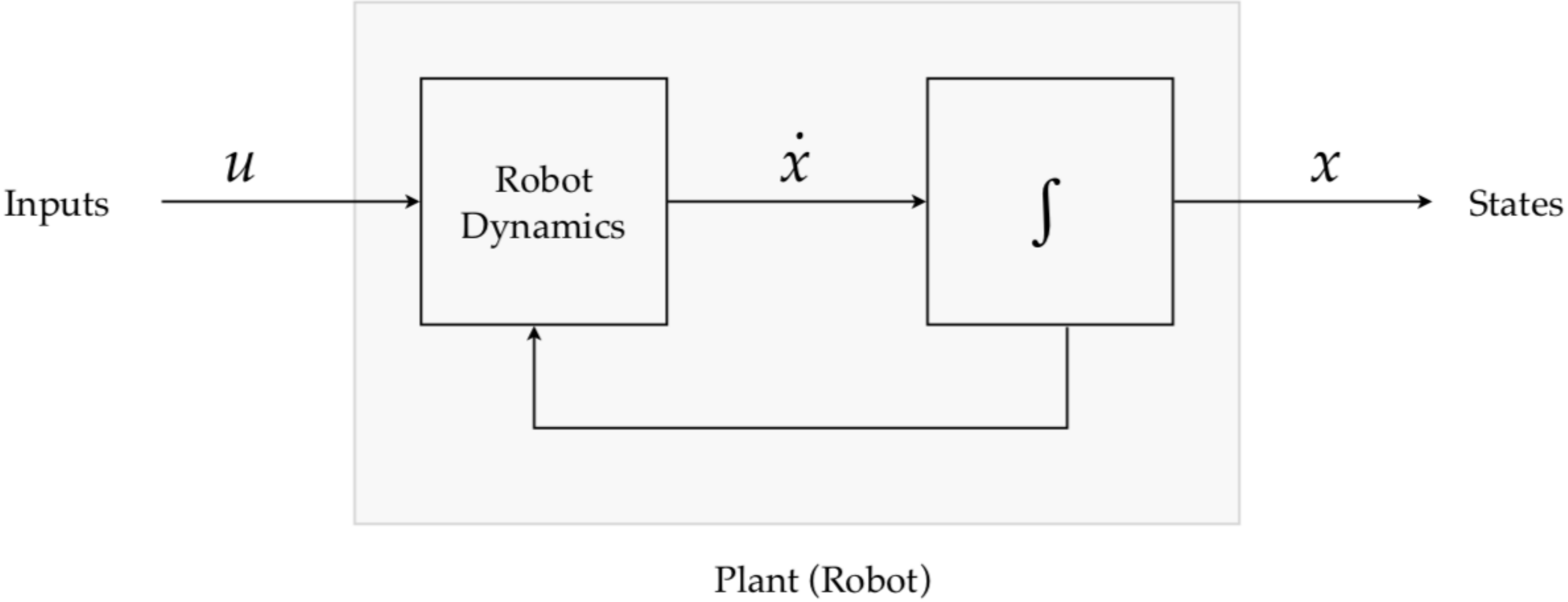
$$\dot{x} = f(x, u, t)$$

$$z = g(x, t)$$

Question

- What would be an example of a system with time-dependent (non-stationary) dynamics?

Stationary Continuous Systems



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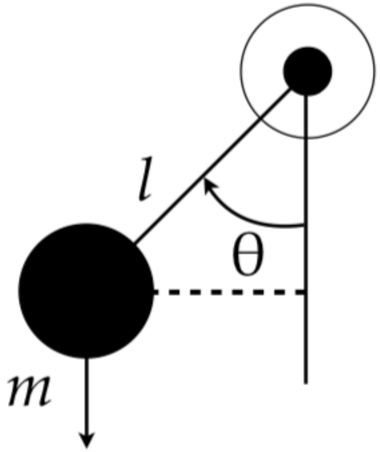
$$z = g(x)$$

Example: Pendulum

Assumptions:

- Point mass *m*
- No friction
- External torque motor



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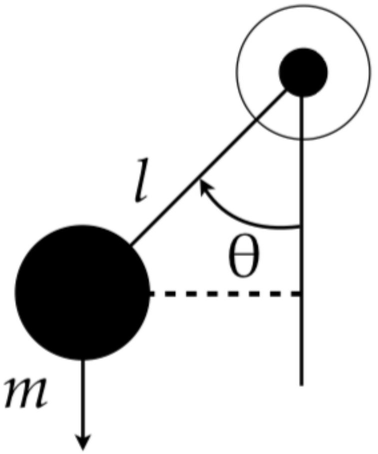

- l: the length of the rod,
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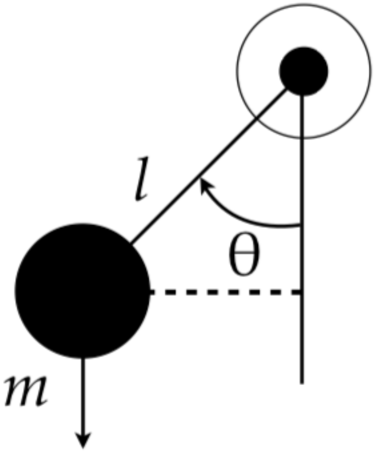

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$$I\ddot{\theta} = -mglsin(\theta) + \tau$$

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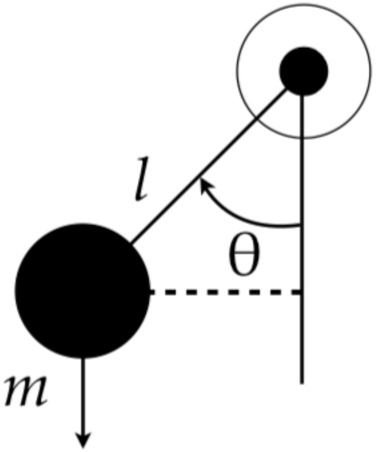

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All torques and forces have to be balanced!

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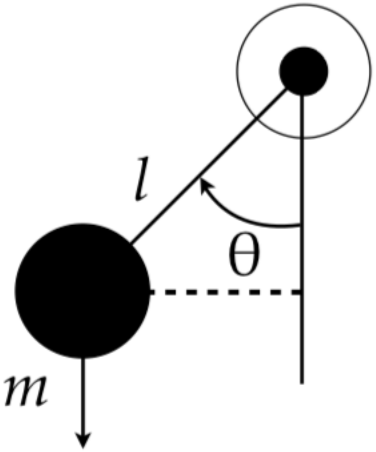

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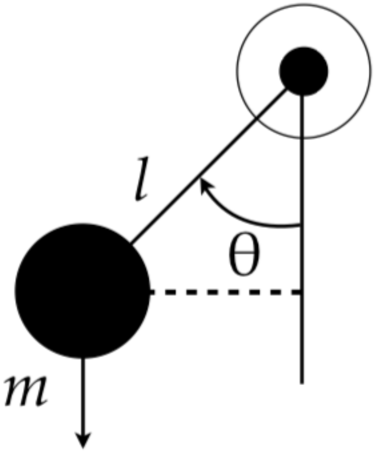

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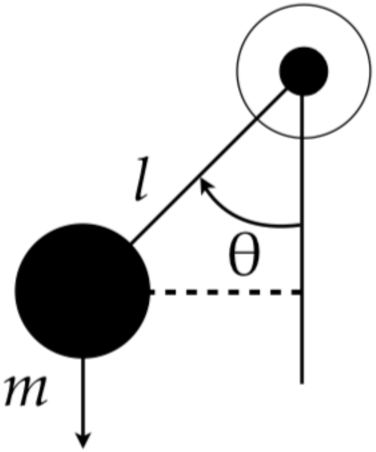

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$$ml^2\ddot{\theta} = -mglsin(\theta) + \tau$$

$$\ddot{\theta} = -\frac{g}{l}sin(\theta) + \frac{\tau}{ml^2}$$

Notation

State: position, velocity

Change of state: acceleration (derivative of state)

Pendulum Equation

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$$x_1 = \dot{\theta}$$

$$x_2 = \theta$$

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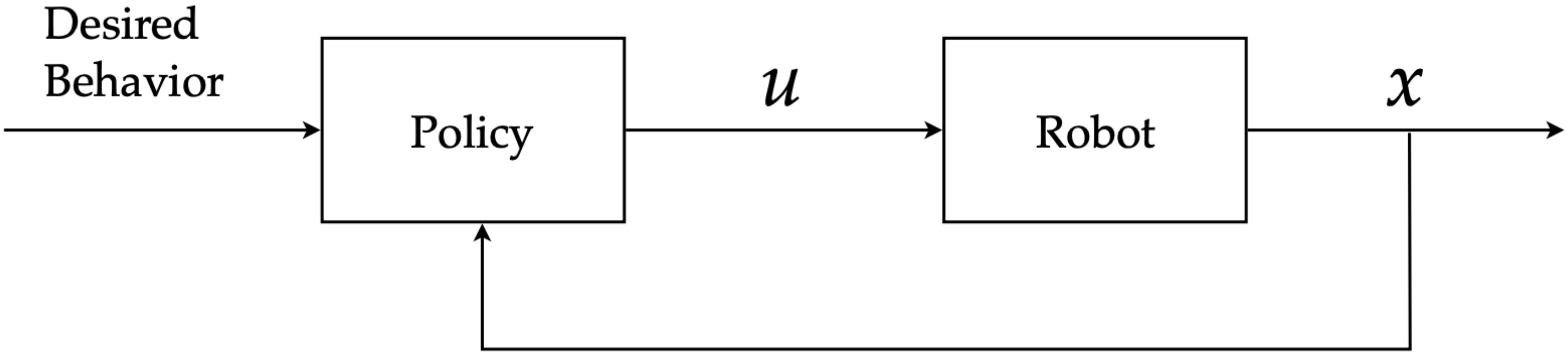
$$z = [0\ 1][x_1\ x_2]^T$$

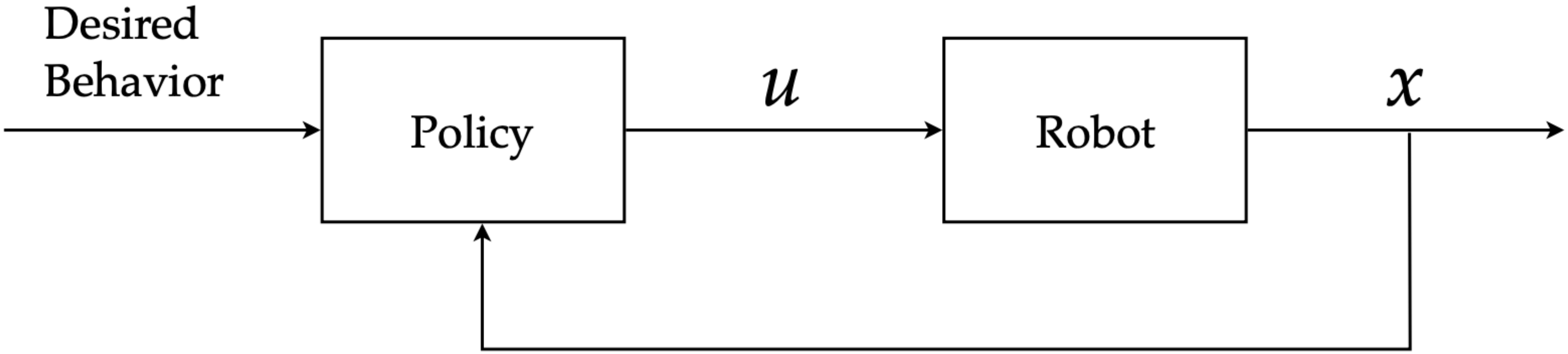
Closed-Loop Control

$$u_t = \pi(x_t, \alpha, t)$$



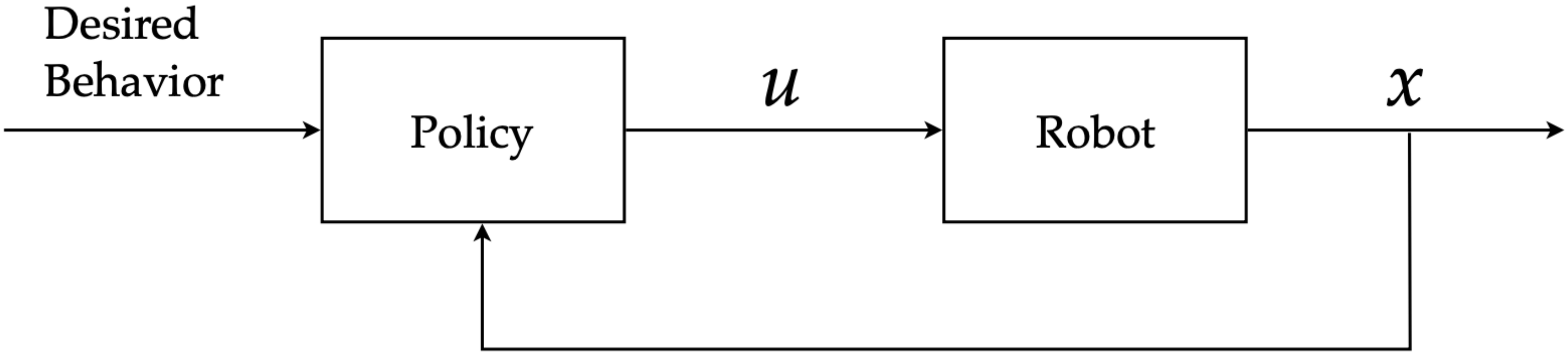
Goal: Compute a Policy

$$u_t = \pi(x_t, \alpha, t)$$



Desired Behavior

$$\min\ J = ||x_T - x_T^*||$$

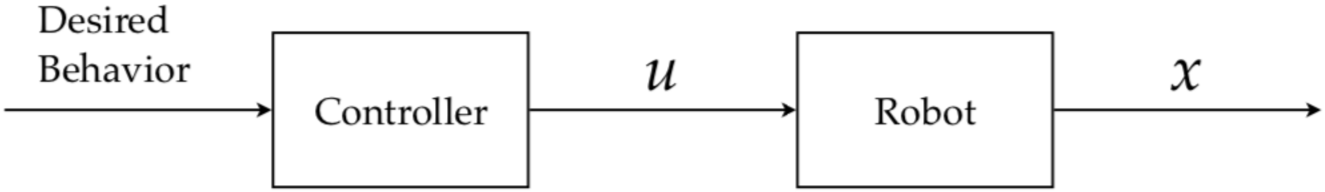


Other desired behaviors:

- Maximize external reward (least specific)
- Track a prespecified trajectory (most specific)

Open-Loop Control

$$u_t = \pi(\alpha, t)$$



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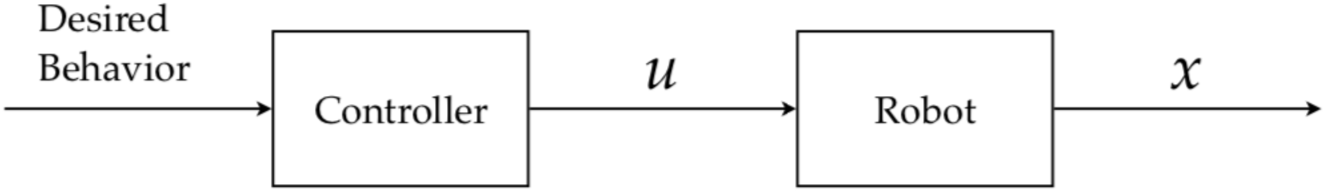

Benefits of open-loop control:
- Fast movement
- Does not require monitoring

Closed-Loop Control

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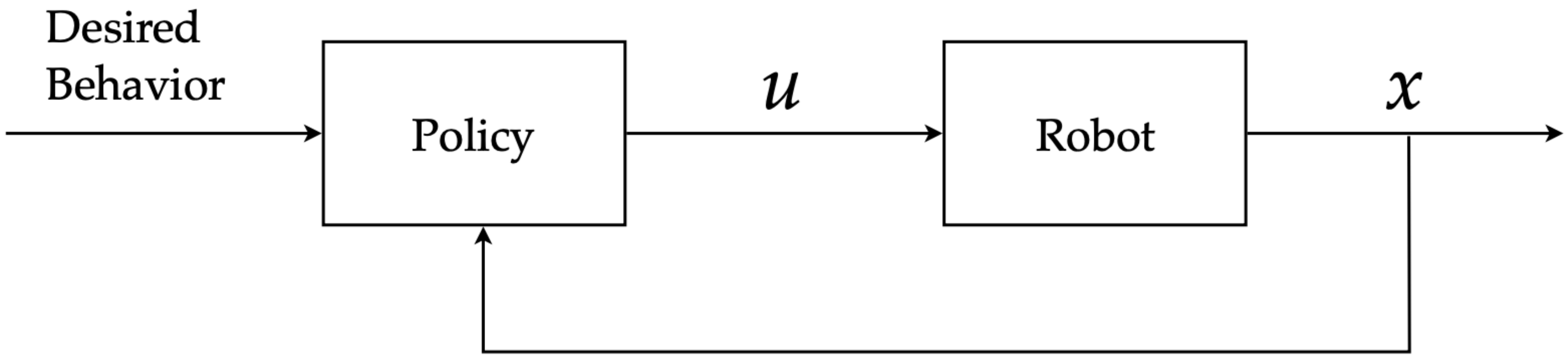
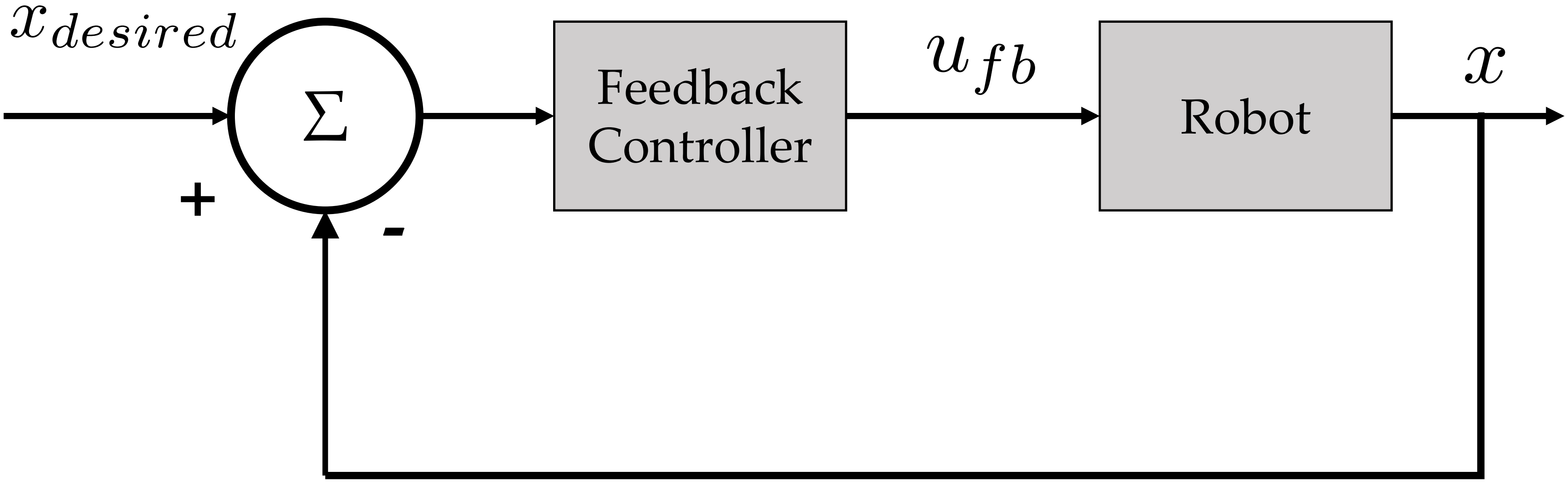
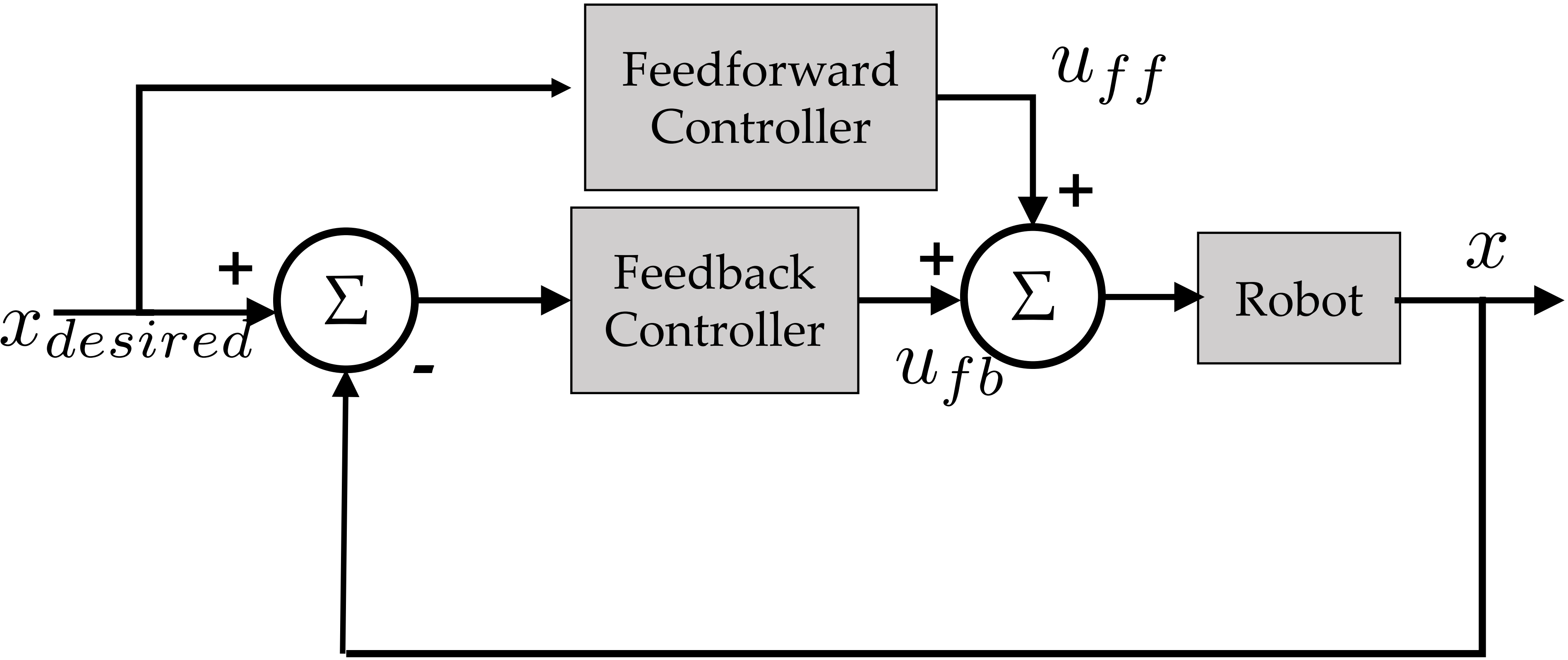


Negative Feedback Control



Negative Feedback & Feedforward Control



Negative Feedback Control

- Based on linear control:
  - Proportional Control ("Position Error")

$$u_P = \pi(x - x_{des}, \alpha, t) = K_P(x_{des}(t) - x(t))$$

  - Derivative Control ("Damping")

$$u_D = \pi(x - x_{des}, \alpha, t) = K_D(\dot{x}_{des}(t) - \dot{x}(t))$$

  - Integral Control ("Steady State Error")

$$u_I(t) = K_I \int_{\tau=0}^{\tau=t} (x_{des}(t) - x(t))dt$$

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Often the state is a vector and the gain is a diagonal matrix

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$$\text{Case 1}: x < x_{des}$$

$$\text{Case 2}: x > x_{des}$$

$$\text{Case 3}: x \approx x_{des}$$

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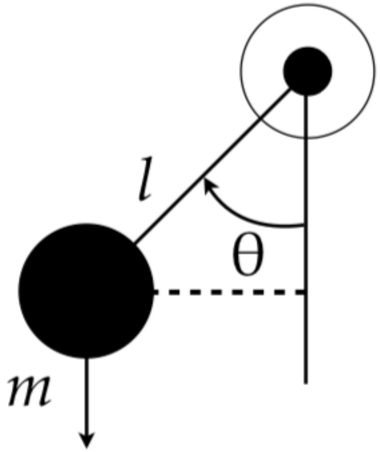

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Proportional-Derivative Control

Desired state: $x_1 = 0$, $x_2 = x_d$

$$\tau = u_p + u_d = K_p(x_d - x_2) + K_d(0 - x_1)$$

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At which position do we reach an equilibrium?

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How do we solve that?

Proportional-Derivative Control

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$$sin(x_2) \sim x_2$$

for small x_2

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$$-gx_2 + \frac{K_p(x_d - x_2)}{ml} = 0$$

$$x_2 = \frac{K_p x_d}{mlg + K_p}$$

Proportional-Derivative Control

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But our goal was $x_2 = x_d$!

If $K_P \rightarrow \infty$

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If $K_P \to \infty$ $x_2 = x_d$

Does not work well in the discrete domain!

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Inserting an integral term solves this problem!