USC Viterbi School of Engineering

Lecture 2: Matrix Algebra Refresher

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CSCI 545 Introduction to Robotics
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Matrix

- A *matrix* of dimensions (m x n), with m and n positive integers, is an array of elements a_{ij} arranged into m *rows* and n columns

$$\boldsymbol{A} = [a_{ij}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Matrix

- If $m = n$, the matrix is said to be *square*; if $m < n$, the matrix has more columns than rows; if $m > n$ the matrix has more rows than columns. Further, if $n = 1$, the matrix is reduced to a column vector $\boldsymbol{a}$ of dimensions (m x 1). The elements a_i are said to be vector components.

Matrix

- A square matrix $\boldsymbol{A}$ of dimensions (n x n) is said to be *upper triangular* if $a_{ij} = 0$ for $i > j$:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

- A square matrix $\boldsymbol{A}$ of dimensions (n x n) is said to be *lower triangular* if $a_{ij} = 0$ for $i < j$:

Matrix

- An $(n \times n)$ square matrix $\boldsymbol{A}$ is said to be *diagonal* if $a_{ij} = 0$ for $i \neq j$, i.e.:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} = \mathrm{diag}\{a_{11}, a_{22}, \dots, a_{nn}\}$$

- Question: How is a matrix called if it has all unit elements in the diagonal:

Matrix

- The *transpose* $\boldsymbol{A^T}$ of a matrix $\boldsymbol{A}$ of dimensions (m x n) is the matrix of dimensions (n x m) which is obtained from the original matrix by interchanging its rows and columns.

$$\boldsymbol{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Question

- What is the transpose of the following matrix?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^T = ?$$

Matrix

- An (n x n) square matrix $\boldsymbol{A}$ is said to be *symmetric* if $\boldsymbol{A}^T = \boldsymbol{A}$ and thus $a_{ij} = a_{ji}$:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

Questions

- Can you give an example of a symmetric matrix?

Questions

- Can you give an example of a symmetric matrix?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Matrix

- An (n x n) square matrix $\boldsymbol{A}$ is said to be skew-*symmetric* if $\boldsymbol{A}^T = -\boldsymbol{A}$ and thus $a_{ij} = -a_{ji}$ for $i \neq j$ and $a_{ii} = 0$, leading to

$$\boldsymbol{A} = \begin{bmatrix} 0 & a_{12} & \dots & a_{1n} \\ -a_{12} & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \dots & 0 \end{bmatrix}$$

Matrix Algebra

$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$ (for matrices that have the same shape)

$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$

$(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$

Matrix Algebra

If $A_s = \frac{1}{2}(A + A^T)$ and

$A_a = \frac{1}{2}(A - A^T)$ then:

$$A = A_s + A_a$$

Example

$$A_s = \frac{1}{2}(A + A^T)$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Example

$$A_s = \frac{1}{2}(A + A^T)$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$A_s = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{bmatrix}$$

Example

$$A_s = \frac{1}{2}(A + A^T) \qquad A_a = \frac{1}{2}(A - A^T)$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \; A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$A_s = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{bmatrix} \; A_a = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

Example

$A_s = \frac{1}{2}(A + A^T)$ $A_a = \frac{1}{2}(A - A^T)$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$A_s = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{bmatrix} A_a = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

$A = A_s + A_a$

Multiplication of Matrices

$$A\,(BC) = (AB)\,C$$

$$A(B+C) = AB + AC$$

$$(A+B)C = AC + BC$$

$$(AB)^T = B^T A^T$$

Multiplication of Matrices

$$AB \;?\; BA$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 8 \\ 14 & 18 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 8 \\ 14 & 18 \end{bmatrix}$$

$$BA = \begin{bmatrix} 8 & 12 \\ 11 & 16 \end{bmatrix}$$

Multiplication of Matrices

$$AB \neq BA$$

Linear Independence

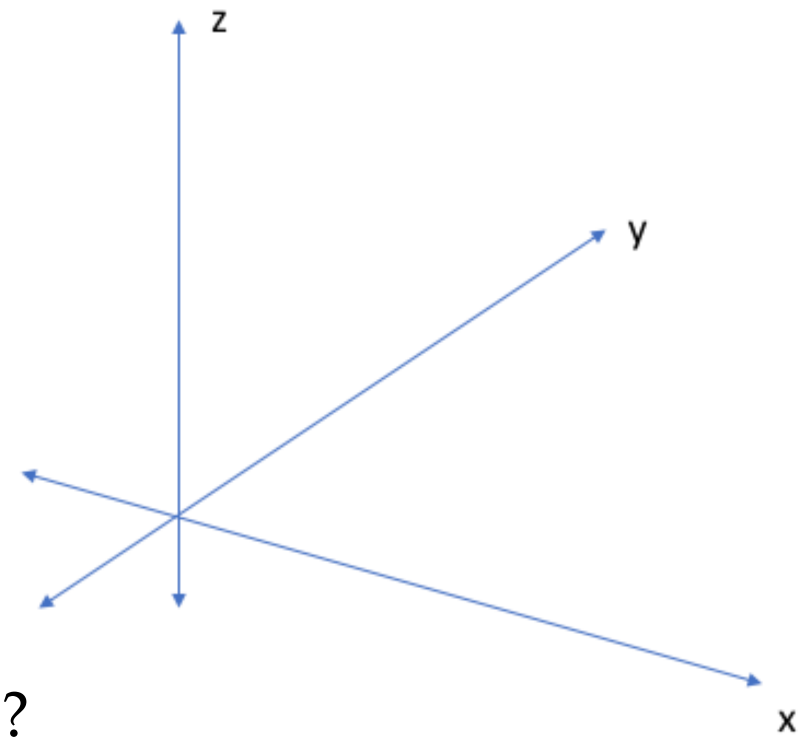
- Rank: This corresponds to the maximal number of linearly independent columns of A.

- A set of vectors is said to be **linearly dependent** if at least one of the vectors in the set can be defined as a linear combination of the others.

- If no vector in the set can be written in this way, then the vectors are said to be **linearly independent**.

Interpretation of Matrices

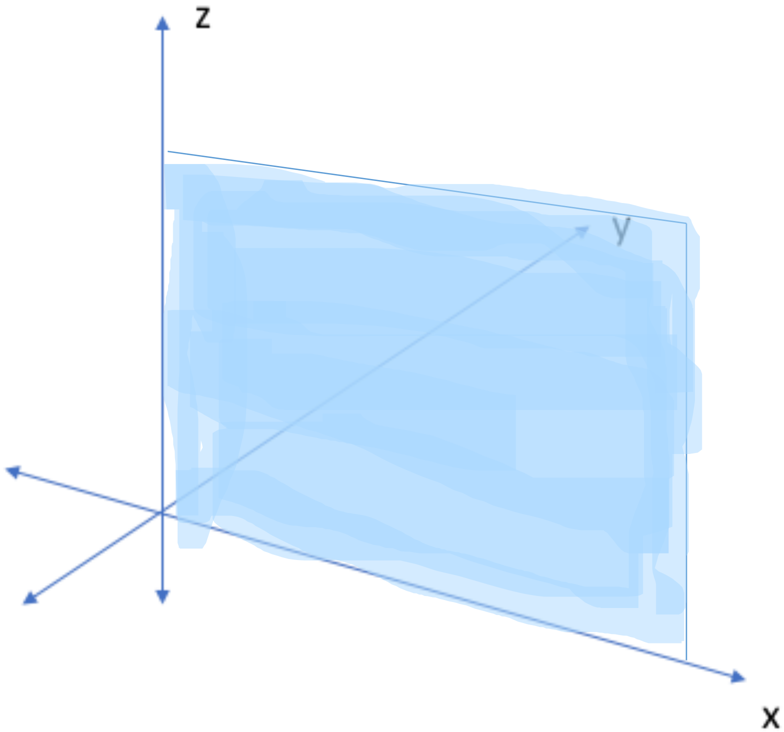
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



What is the rank of this matrix?

Interpretation of Matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$\det(\boldsymbol{A}) = 0$

What is the rank of this matrix?

Inverse Matrices

$$A^{-1}A = I$$

Matrix Transformations

Vector ***w*** is transformed to a new space

$$Aw = v$$

Vector ***w*** is transformed back to the old space

$$w = A^{-1}v$$

But what if ***A*** is rank deficient?

Orthogonal Matrices

If $\boldsymbol{A}$ is a square matrix, and:

$$A^T = A^{-1}$$

then the matrix is called **orthogonal**.

Properties of Inverse Matrices

If ***A*** and ***B*** are square invertible matrices:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Matrix Derivatives

The *derivative* of an $(m \times n)$ matrix $\boldsymbol{A}(t)$, whose elements $a_{ij}(t)$ are differentiable functions, is the matrix

$$\dot{A}(t) = \frac{d}{dt}A(t) = \left[\frac{d}{dt}a_{ij}(t)\right]_{i=1,\ldots m, j=1,\ldots,n}$$

Example

$$A = \begin{bmatrix} t & t^2 \\ 7t & 1 \end{bmatrix} \dot{A}(t) = ?$$

Example

$$\dot{A}(t) = \begin{bmatrix} 1 & 2t \\ 7 & 0 \end{bmatrix}$$

Gradient of a Function

$$\nabla_x f(x) = \left(\frac{\partial f(x)}{\partial x}\right) = \left[\frac{\partial f(x)}{\partial x_1} \quad \frac{\partial f(x)}{\partial x_2} \quad \cdots \quad \frac{\partial f(x)}{\partial x_n}\right]$$

Example

$$f(x_1, x_2) = x_1 * x_2 + x_1^2$$

$$\nabla_x f(x) = ?$$

Example

$$f(x_1, x_2) = x_1 * x_2 + x_1^2$$

$$\nabla_x f(x) = [\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}]$$

$$= [x_2 + 2x_1, x_1]$$

Gradient of a Function

$$\dot{f}(x) = \frac{d}{dt} f(x) = \frac{\partial f}{\partial x} \dot{x} = \nabla_x f(x) \dot{x}$$

Example

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$x_1(t) = 3t$$
$$x_2(t) = 4t^2$$

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$$\dot{f}(x) = \frac{d}{dt} f(x)$$

Example

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$x_1(t) = 3t$$
$$x_2(t) = 4t^2$$

$$\begin{aligned}
\dot{f}(x) &= \frac{d}{dt} f(x) \\
&= \frac{\partial f}{\partial x} \dot{x} \\
&= \bigtriangledown_x f(x) \dot{x} \\
&= (2x_1, 2x_2)^T (3, 8t) \\
&= (6t, 8t^2)^T (3, 8t) \\
&= 18t + 64t^3
\end{aligned}$$

Jacobian Matrix

$$J_g(x) = \frac{g(x)}{\partial x} = \begin{bmatrix} \frac{\partial g_1(x)}{\partial x} \\ \frac{\partial g_2(x)}{\partial x} \\ \vdots \\ \frac{\partial g_m(x)}{\partial x} \end{bmatrix}$$

Example

$$g = [g_1, g_2]$$

$$g_1(x_1, x_2) = x_1 * x_2 + x_1^2$$

$$g_2(x_1, x_2) = x_1^4$$

Example

$$g = [g_1, g_2]$$

$$g_1(x_1, x_2) = x_1 * x_2 + x_1^2$$

$$g_2(x_1, x_2) = x_1^4$$

$$\frac{\partial g_1}{\partial x} = [x_2 + 2x_1, x_1]$$

$$\frac{\partial g_2}{\partial x} = [4 * x_1^3, 0]$$

Example

$$g = [g_1, g_2]$$

$$g_1(x_1, x_2) = x_1 * x_2 + x_1^2$$

$$g_2(x_1, x_2) = x_1^4$$

$$\frac{\partial g_1}{\partial x} = [x_2 + 2x_1, x_1]$$

$$\frac{\partial g_2}{\partial x} = [4 * x_1^3, 0]$$

$$J = \begin{bmatrix} x_2 + 2x_1 & x_1 \\ 4x_1^3 & 0 \end{bmatrix}$$

Time Derivative of Vector Function

$$\dot{g}(x) = \frac{d}{dt} g(x(t)) = \frac{\partial g}{\partial x} \dot{x} = J_g(x) \dot{x}$$

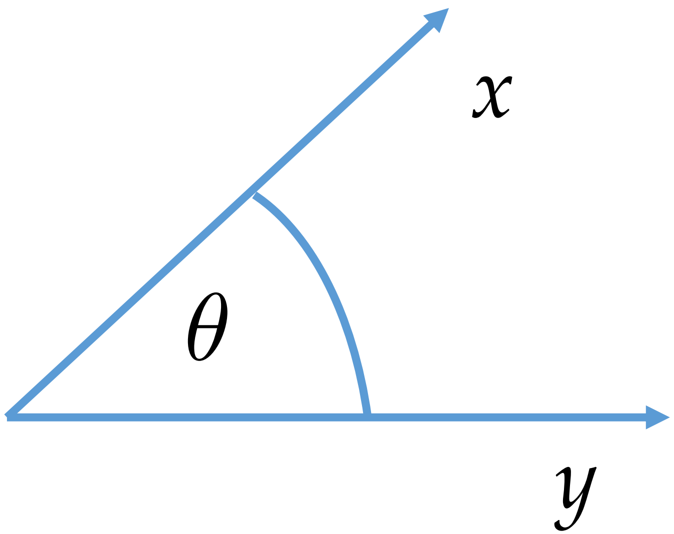
Inner Product

Given n vectors x_i of dimensions (m x 1), the **inner product** is defined as :

$$< x, y >= x_1 y_1 + x_2 y_2 + ... + x_m y_m = x^T y = y^T x$$

Inner Product: Geometric Interpretation

$$x^T y = ||x|| \, ||y|| cos(\theta)$$



What is the geometric interpretation of the dot product if the vector y has length = 1?

Orthogonal vectors

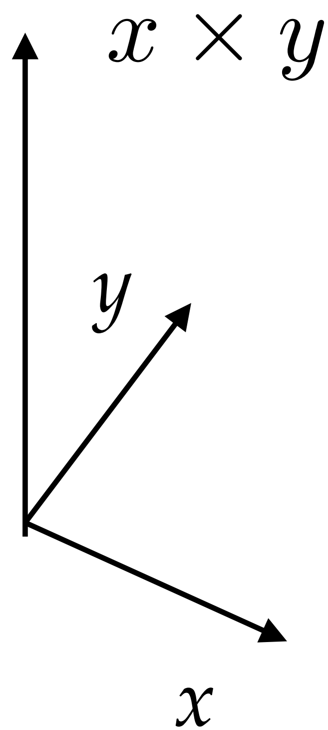
- Two vectors are said to be ***orthogonal*** when their scalar product is 0:

$$x^T y = 0$$

Cross Product

$$x \times y = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_3 - x_2y_1 \end{bmatrix}$$

Cross Product



Question: How do you get the cross-product of two vectors to be 0?

Cross Product Operator

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Cross Product Operator

$$x \times y = S(x)y = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} y$$

Eigenvalues and Eigenvectors

- There are some vector ***u***, that if you multiply with ***A*** they either stretch or become shorter, but they state in the same direction of motion.

$$Au - \lambda u = 0$$

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Eigenvalues and Eigenvectors

- There are some vector ***u***, that if you multiply with ***A*** they either stretch or become shorter, but they state in the same direction of motion.

$$Au - \lambda u = 0$$

$$(A - \lambda I)u = 0$$

$$det(A - \lambda I) = 0$$

Example

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$det(\lambda I - A) = det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix})$$

Example

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\begin{aligned} det(\lambda I - A) &= det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}) \\ &= det(\begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{bmatrix}) \\ &= \lambda^2 + 3\lambda + 2 = 0 \end{aligned}$$

Example

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\begin{aligned} det(\lambda I - A) &= det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}) \\ &= det(\begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{bmatrix}) \\ &= \lambda^2 + 3\lambda + 2 = 0 \end{aligned}$$

$$\lambda_1 = -2, \lambda_2 = -1$$

Positive Definiteness

- The matrix ***A*** is positive definite iff:

$$x^T A x > 0, \forall x \neq 0$$

$$x^T A x = 0, \forall x = 0$$

Example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$[x_1, x_2] \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} [x_1, x_2]^T = [x_1, 2x_2][x_1, x_2]^T$$

$$= x_1^2 + 2x_2^2$$