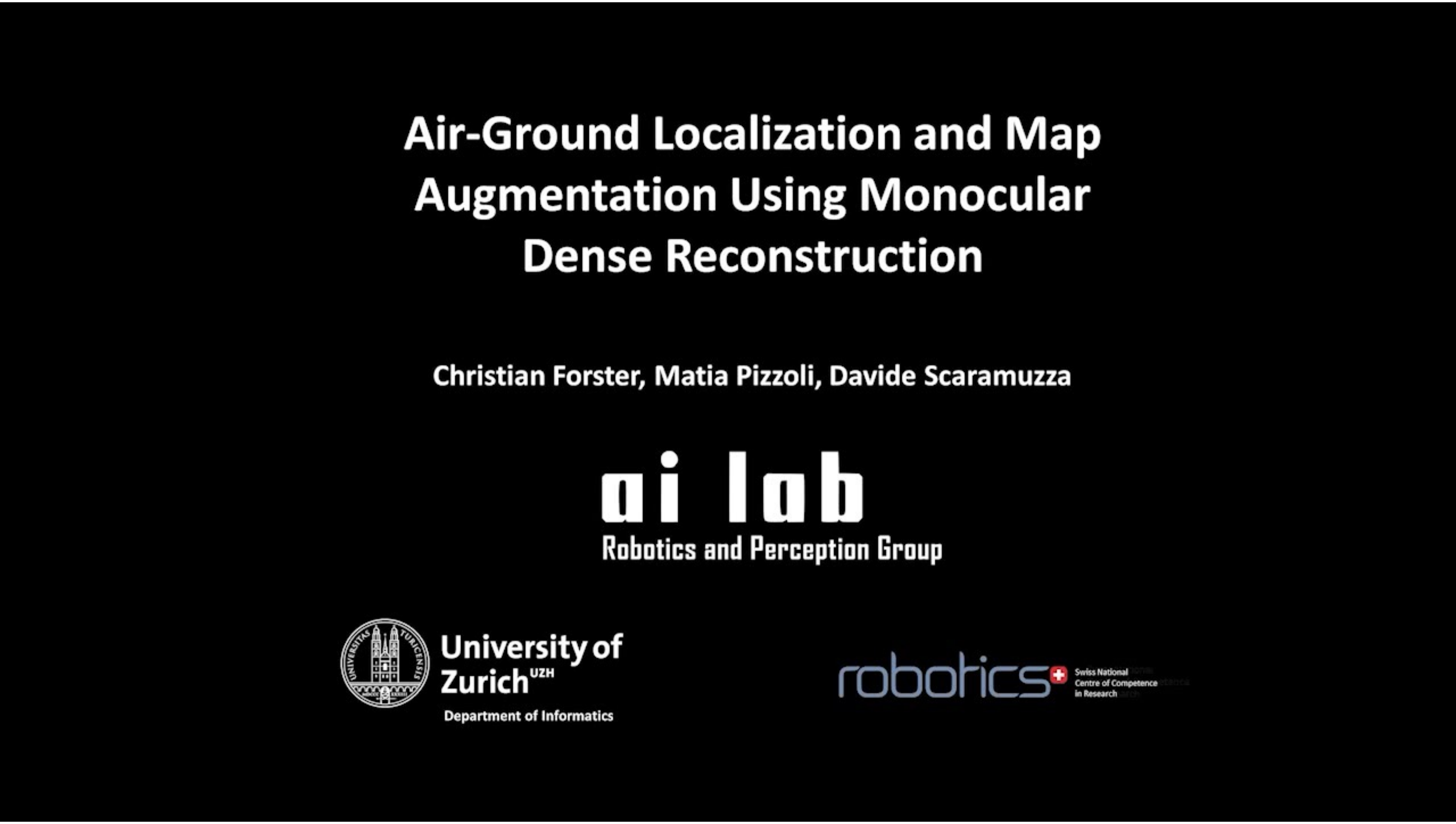
Kalman Filters

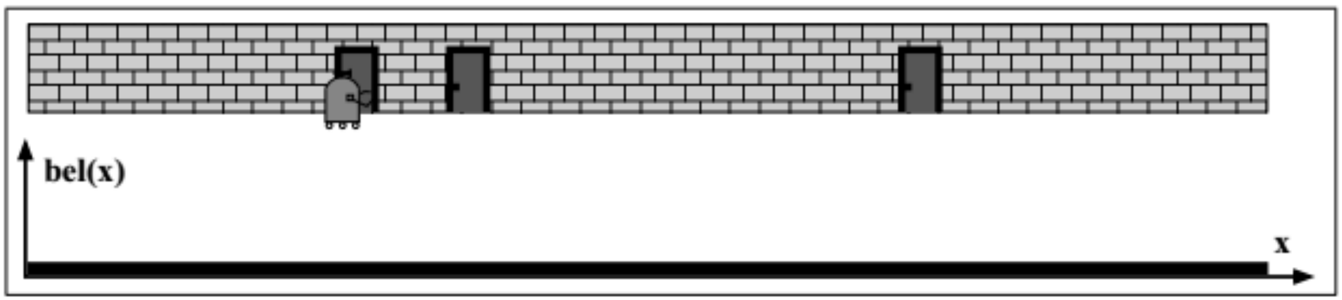
CSCI 545 Introduction to Robotics
Instructor: Stefanos Nikolaidis
Resources: 16.322 Stochastic Estimation and Control, MIT

State Estimation

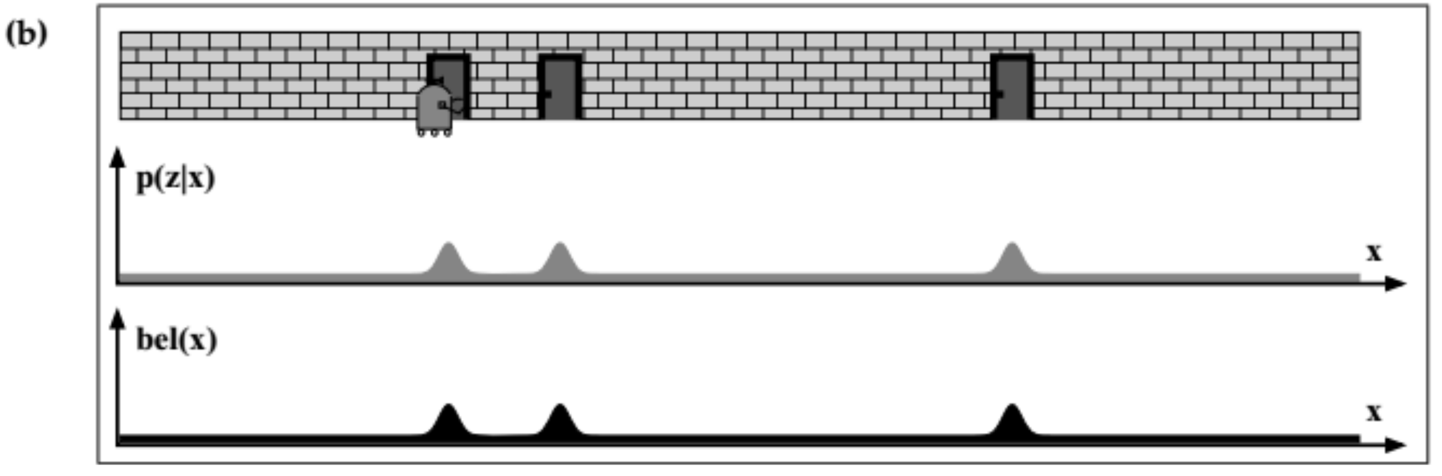


Localization Example

(a)

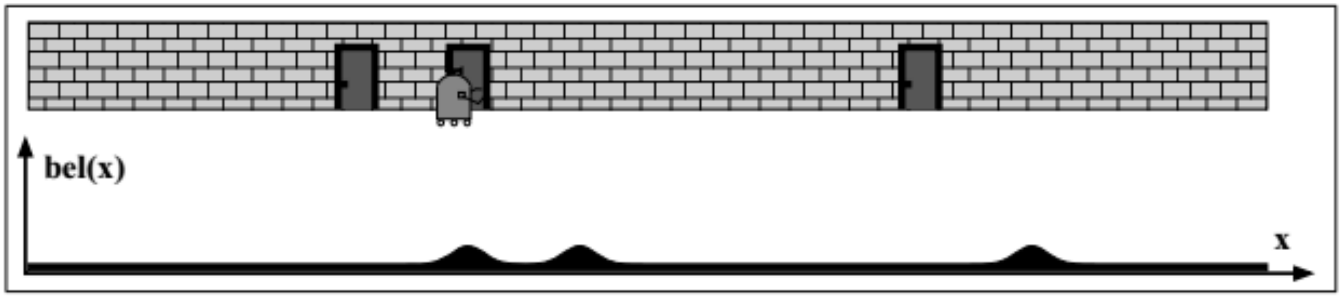


Localization Example



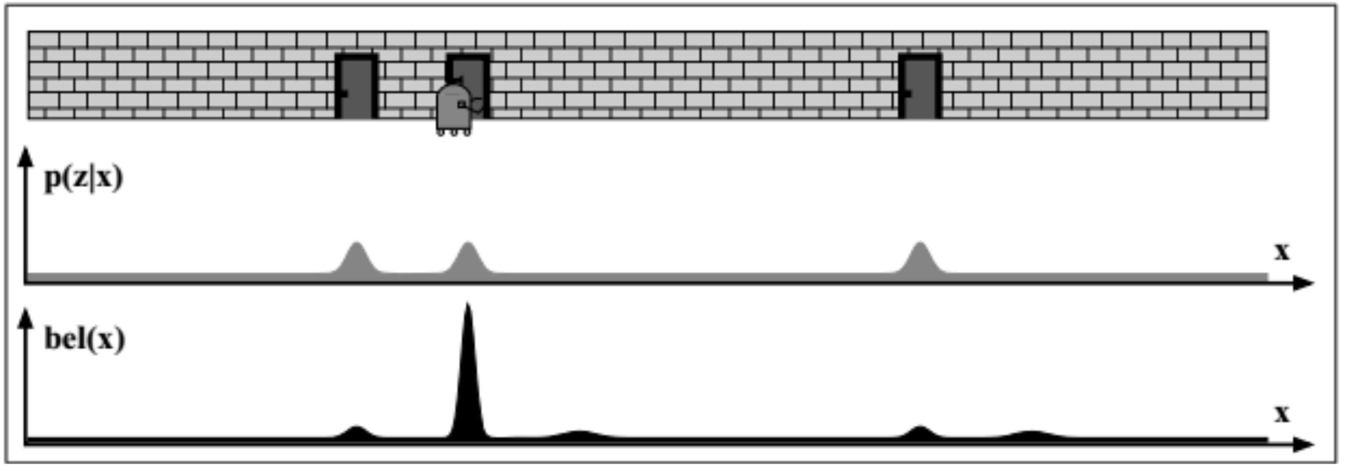
Localization Example

(c)

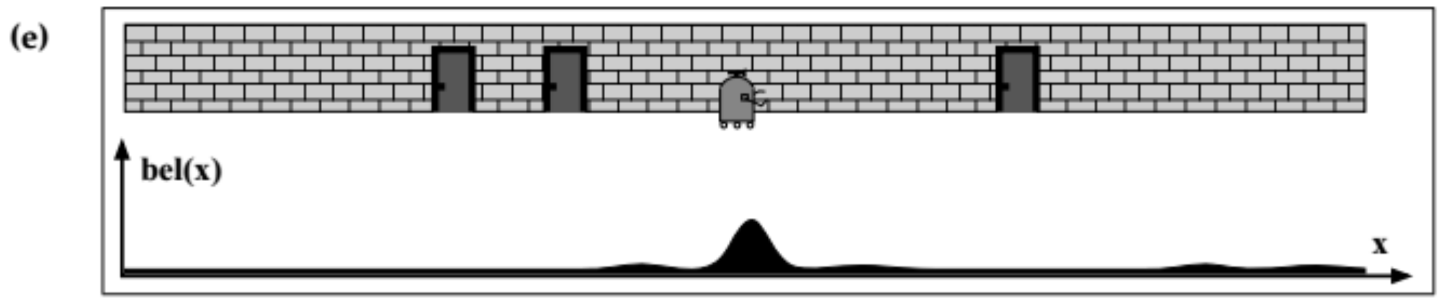


Localization Example

(d)



Localization Example



What about continuous states?

- We will assume that the dynamics model is linear.

$$x_t = Ax_{t-1} + Bu_t + \epsilon$$

$$x_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{bmatrix} \qquad u_t = \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ \vdots \\ u_{m,t} \end{bmatrix}$$

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A is an *n* x *n* matrix, *B* is *n* x *m*

$$\epsilon \sim N(0, R)$$

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$$p(x_t|u_t, x_{t-1}) =$$

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$$p(x_t|u_t, x_{t-1}) = \det(2\pi R)^{-\frac{1}{2}} \exp\{-\frac{1}{2}(x_t - Ax_{t-1} - Bu_t)^T R^{-1}(x_t - Ax_{t-1} - Bu_t)\}$$

Sensor Model

$$z_t = Cx_t + \delta$$

$$z_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \\ \vdots \\ z_{k,t} \end{bmatrix}$$

C is an *k* x *n* matrix

$$\delta \sim N(0, Q)$$

Sensor Model

$$z_t = Cx_t + \delta$$

$$z_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \\ \vdots \\ z_{k,t} \end{bmatrix}$$

$$p(z_t|x_t) = \det(2\pi Q)^{-\frac{1}{2}} \exp\{-\frac{1}{2}(z_t - Cx_t)^T Q^{-1}(z_t - Cx_t)\}$$

How is information propagated?

- Initial belief of x_0:

$$b(x_0) = \det(2\pi\Sigma_0)^{-\frac{1}{2}} \exp\{-\frac{1}{2}(x_0 - \mu_0)^T \Sigma_0^{-1}(x_0 - \mu_0)\}$$

- Propagation of mean

$$\begin{aligned} \mu_t &= E[Ax_{t-1} + Bu_t + \epsilon] \\ &= E[Ax_{t-1}] + E[Bu_t] + E[\epsilon] \\ &= AE[x_{t-1}] + Bu_t + E[\epsilon] \\ &= A\mu_{t-1} + Bu_t \end{aligned}$$

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- Propagation of Covariance

$$\Sigma_t = E[(x_t - \mu_t)(x_t - \mu_t)^T]$$

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$$\begin{aligned} x_t - \mu_t &= Ax_{t-1} + Bu_t + \epsilon - \mu_t \\ &= Ax_{t-1} + Bu_t + \epsilon - A\mu_{t-1} - Bu_t \\ &= A(x_{t-1} - \mu_{t-1}) + \epsilon \end{aligned}$$

Propagation of covariance

$$
\begin{aligned}
x_t - \mu_t &= Ax_{t-1} + Bu_t + \epsilon - \mu_t \\
&= Ax_{t-1} + Bu_t + \epsilon - A\mu_{t-1} - Bu_t \\
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\end{aligned}
$$

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Propagation of covariance

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&= E[(A(x_{t-1} - \mu_{t-1}) + \epsilon)((x_{t-1} - \mu_{t-1})^T A^T + \epsilon^T)] \\
&= E[A(x_{t-1} - \mu_{t-1})(x_{t-1} - \mu_{t-1} -)^T A^T \\
&+ A(x_{t-1} - \mu_{t-1})\epsilon^T + \epsilon(x_{t-1} - \mu_{t-1})^T A^T + \epsilon\epsilon^T] \\
&= E[A(x_{t-1} - \mu_{t-1})(x_{t-1} - \mu_{t-1})^T A^T] \\
&+ E[A(x_{t-1} - \mu_{t-1})\epsilon^T] + E[\epsilon(x_{t-1} - \mu_{t-1})^T A^T] + E[\epsilon\epsilon^T]
\end{aligned}
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&= E[(A(x_{t-1} - \mu_{t-1}) + \epsilon)((x_{t-1} - \mu_{t-1})^T A^T + \epsilon^T)]
\end{aligned}
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Propagation of covariance

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&= E[(A(x_{t-1} - \mu_{t-1}) + \epsilon)((x_{t-1} - \mu_{t-1})^T A^T + \epsilon^T)] \\
&= E[A(x_{t-1} - \mu_{t-1})(x_{t-1} - \mu_{t-1})^T A^T \\
&+ A(x_{t-1} - \mu_{t-1})\epsilon^T + \epsilon(x_{t-1} - \mu_{t-1})^T A^T + \epsilon\epsilon^T] \\
&= E[A(x_{t-1} - \mu_{t-1})(x_{t-1} - \mu_{t-1})^T A^T] \\
&+ E[A(x_{t-1} - \mu_{t-1})\epsilon^T] + E[\epsilon(x_{t-1} - \mu_{t-1})^T A^T] + E[\epsilon\epsilon^T]
\end{aligned}
$$

Propagation of covariance

$$
\begin{aligned}
E[A(x_{t-1} - \mu_{t-1})(x_{t-1} - \mu_{t-1})^T A^T] &= AE[(x_{t-1} - \mu_{t-1})(x_{t-1} - \mu_{t-1})^T]A^T \\
&= A\Sigma_{t-1}A^T
\end{aligned}
$$

Propagation of covariance

$$E[A(x_{t-1} - \mu_{t-1})(x_{t-1} - \mu_{t-1})^T A^T] = AE[(x_{t-1} - \mu_{t-1})(x_{t-1} - \mu_{t-1})^T]A^T$$
$$= A\Sigma_{t-1}A^T$$

$$E[A(x_{t-1} - \mu_{t-1})\epsilon^T] = 0$$

Propagation of covariance

$$E[A(x_{t-1} - \mu_{t-1})(x_{t-1} - \mu_{t-1})^T A^T] = AE[(x_{t-1} - \mu_{t-1})(x_{t-1} - \mu_{t-1})^T]A^T$$
$$= A\Sigma_{t-1}A^T$$

$$E[A(x_{t-1} - \mu_{t-1})\epsilon^T] = 0$$

$$E[\epsilon(x_{t-1} - \mu_{t-1})^T A^T] = 0$$

$$E[\epsilon\epsilon^T] = R$$

Propagation of covariance

$$\begin{aligned}\Sigma_t &= E[(x_t - \mu_t)(x_t - \mu_t)^T] \\ &= A\Sigma_{t-1}A^T + R\end{aligned}$$

Propagation of mean and covariance

$$\mu_t = A\mu_{t-1} + Bu_t$$

$$\Sigma_t = A\Sigma_{t-1}A^T + R$$

Kalman Filter

Algorithm Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$**):**

$\bar{\mu}_t = A_t \, \mu_{t-1} + B_t \, u_t$

$\bar{\Sigma}_t = A_t \, \Sigma_{t-1} \, A_t^T + R_t$

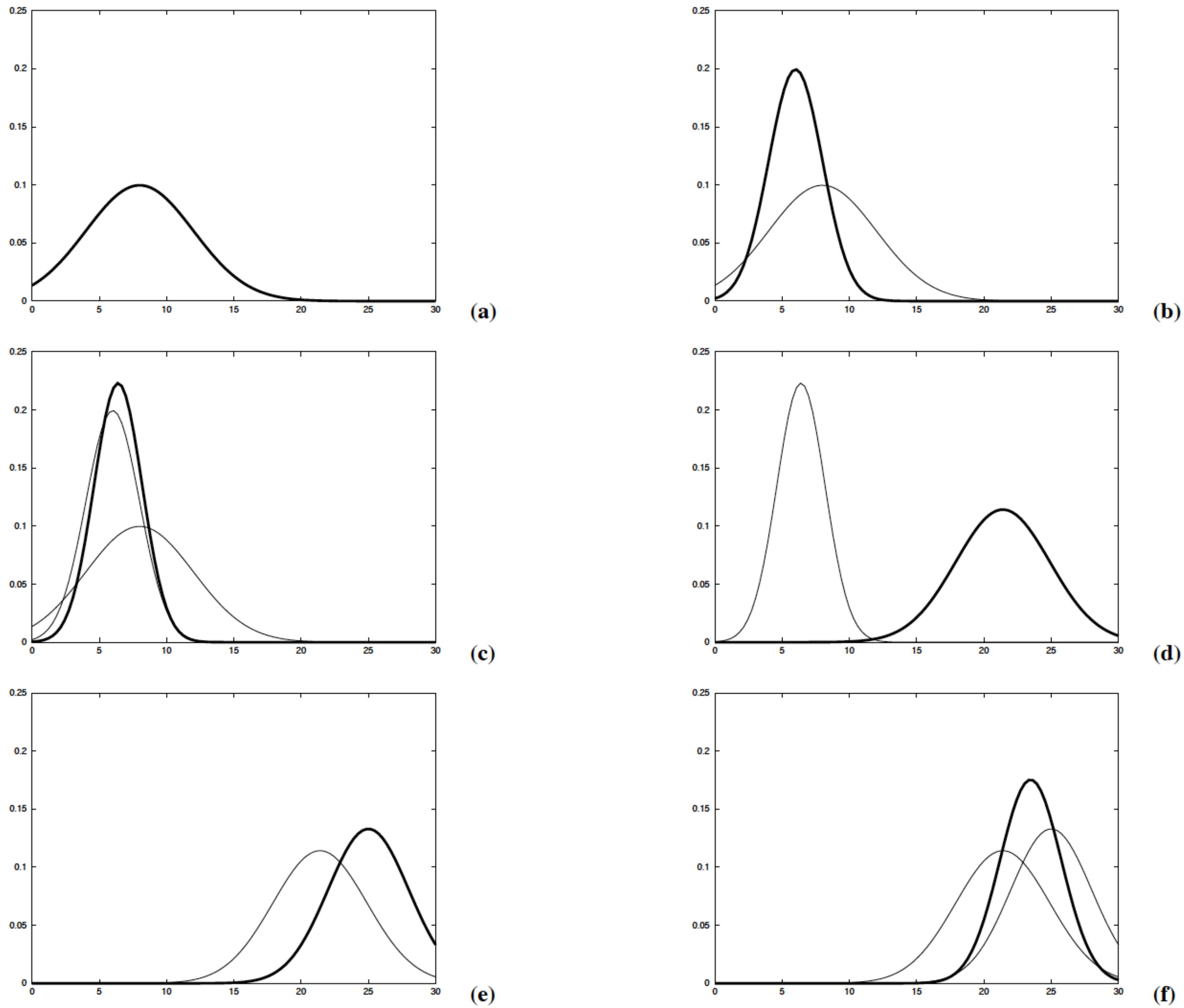
$K_t = \bar{\Sigma}_t \, C_t^T (C_t \, \bar{\Sigma}_t \, C_t^T + Q_t)^{-1}$

$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \, \bar{\mu}_t)$

$\Sigma_t = (I - K_t \, C_t) \, \bar{\Sigma}_t$

return μ_t, Σ_t

Kalman Filter



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return μ_t, Σ_t

K_t = Kalman Gain

Q_t = Measurement Noise (Covariance)

If $Q_t \to \infty$, how does K_t change?

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$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \, \bar{\mu}_t)$
$\Sigma_t = (I - K_t \, C_t) \, \bar{\Sigma}_t$
return μ_t, Σ_t

K_t = Kalman Gain
Q_t = Measurement Noise (Covariance)

What about $Q_t \to 0$?

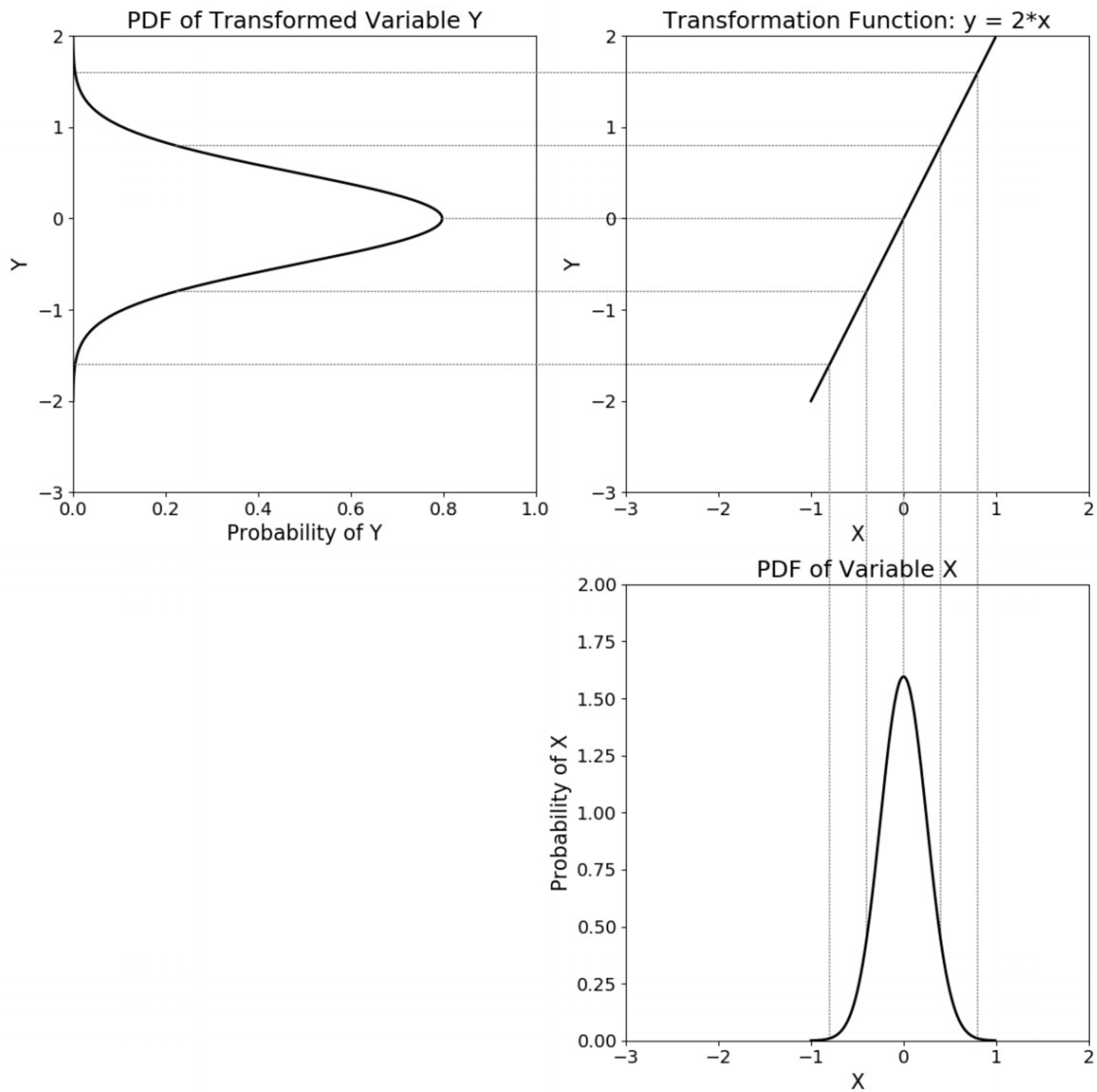
Non-Linear Systems

The linear system assumptions are rarely met in practice!

$$x_t = g(u_t, x_{t-1}) + \epsilon$$

$$z_t = h(x_t) + \delta$$

Linearity Assumption



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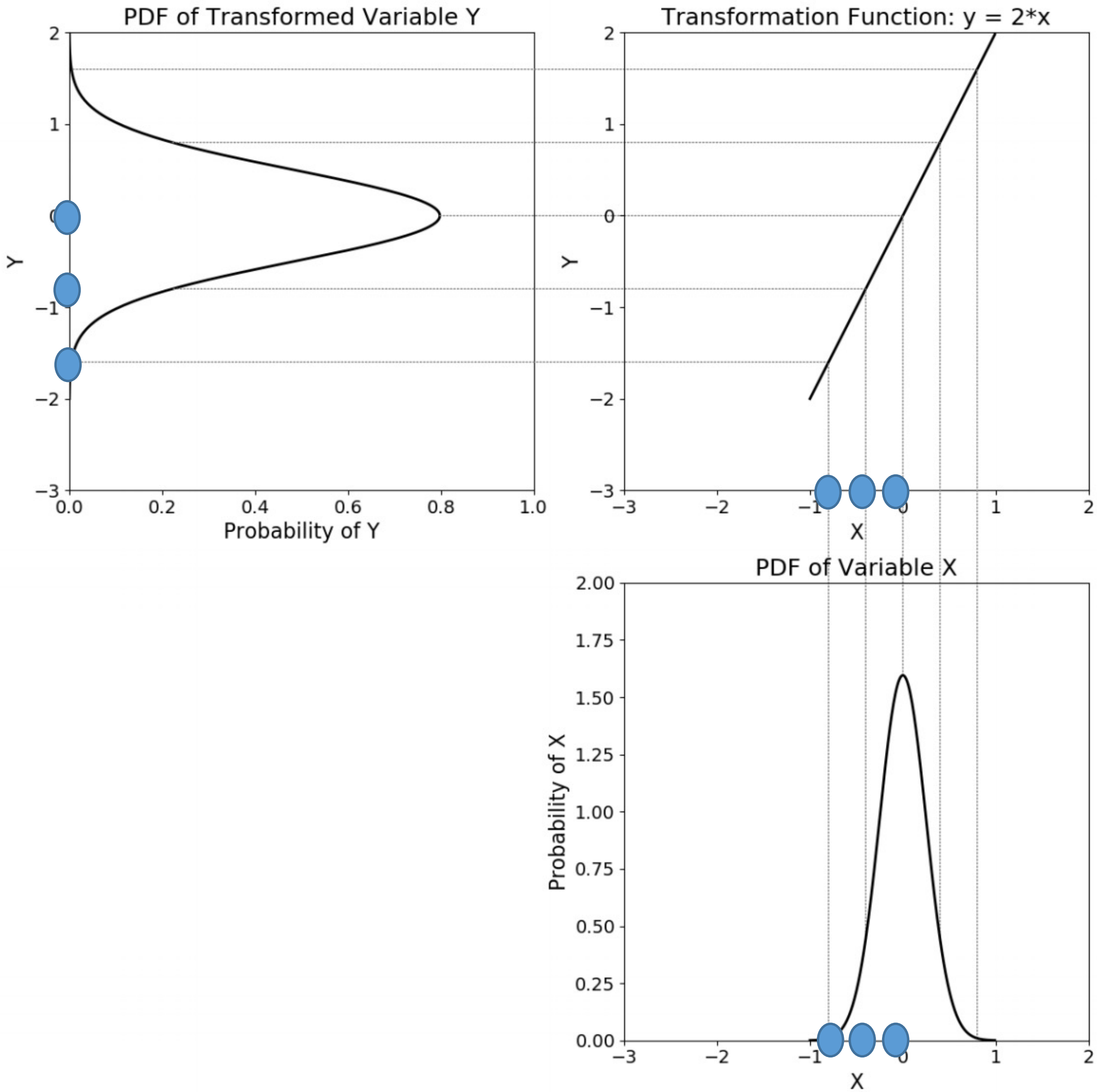

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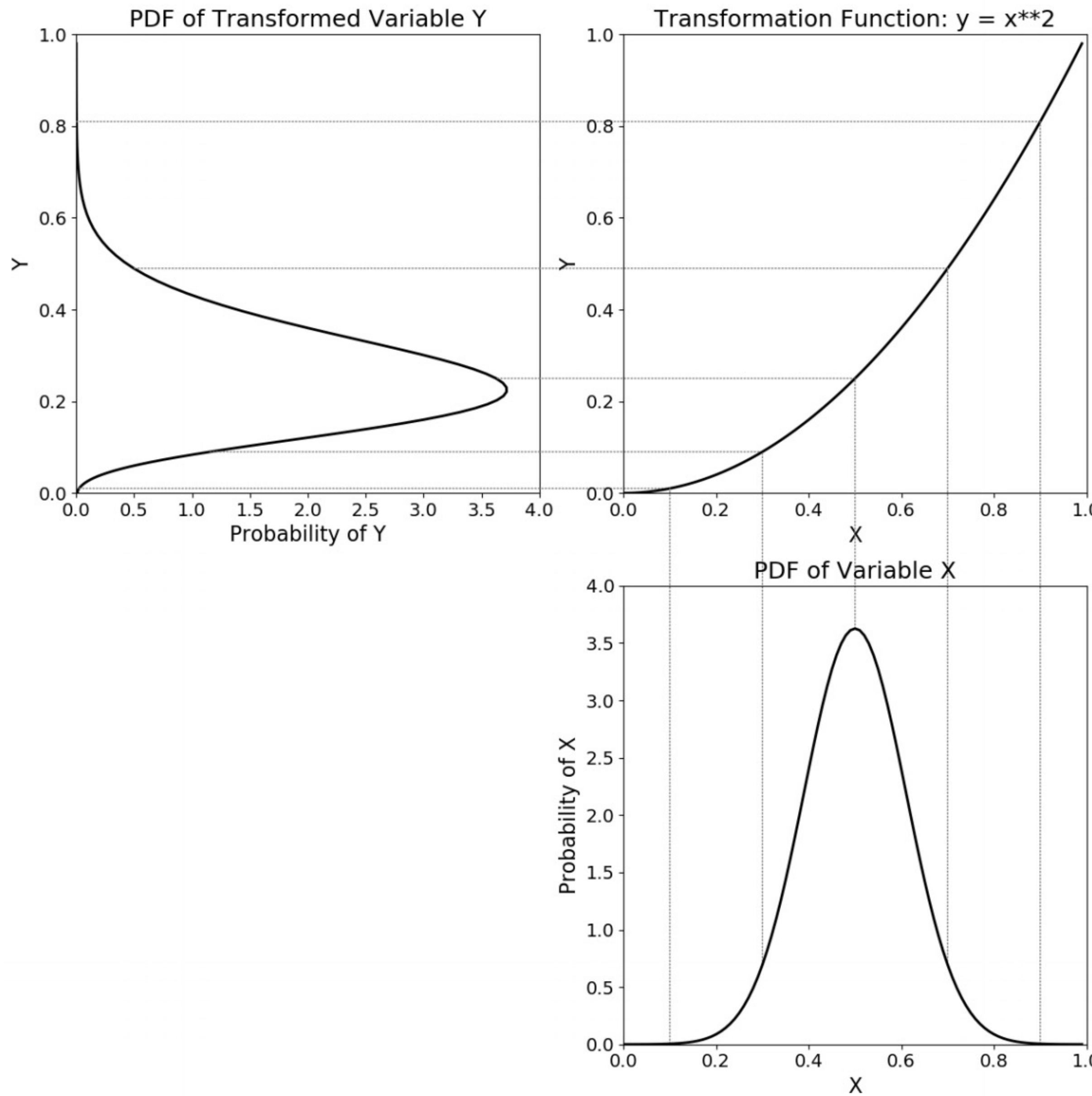
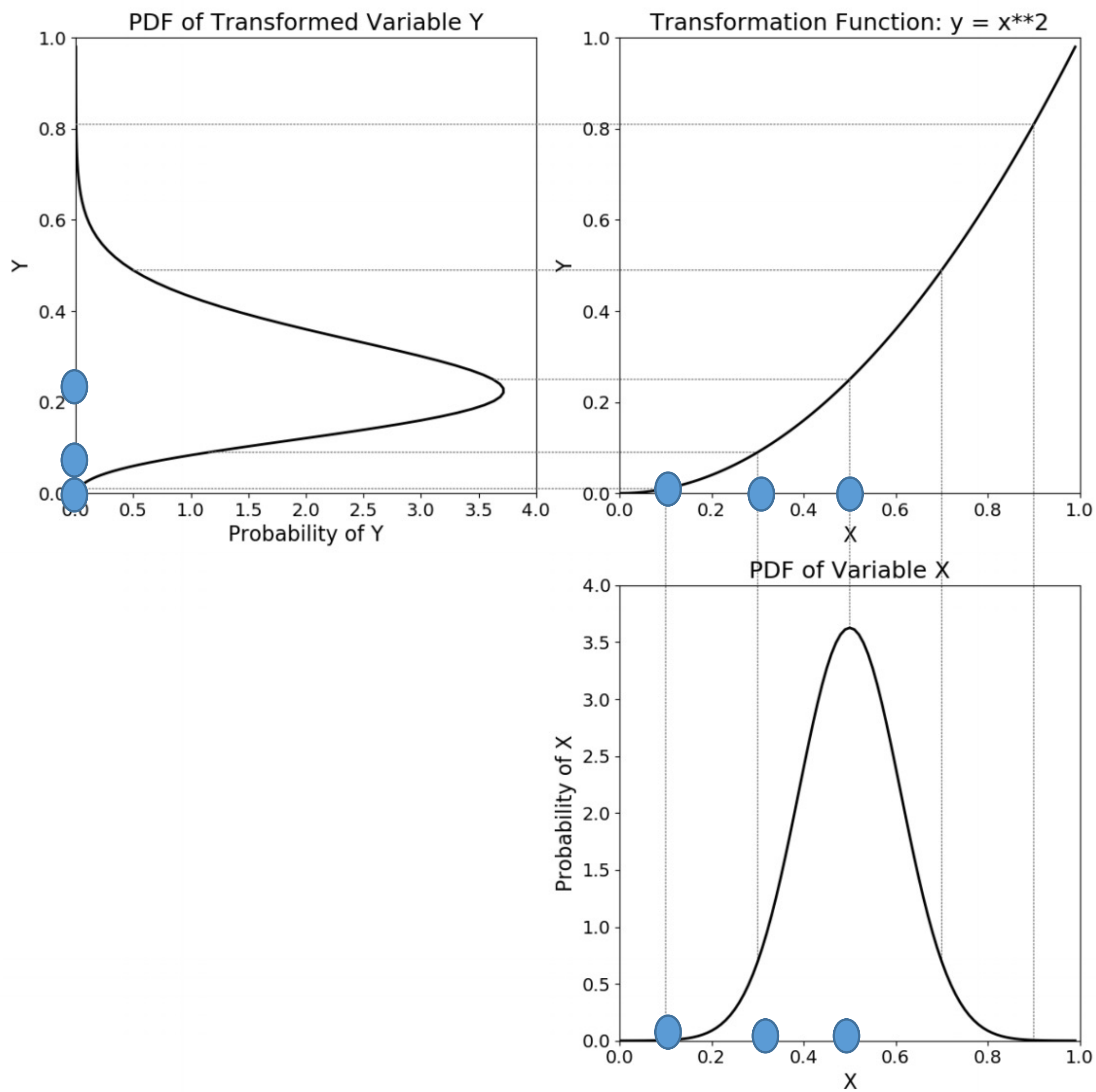


Linearity Assumption

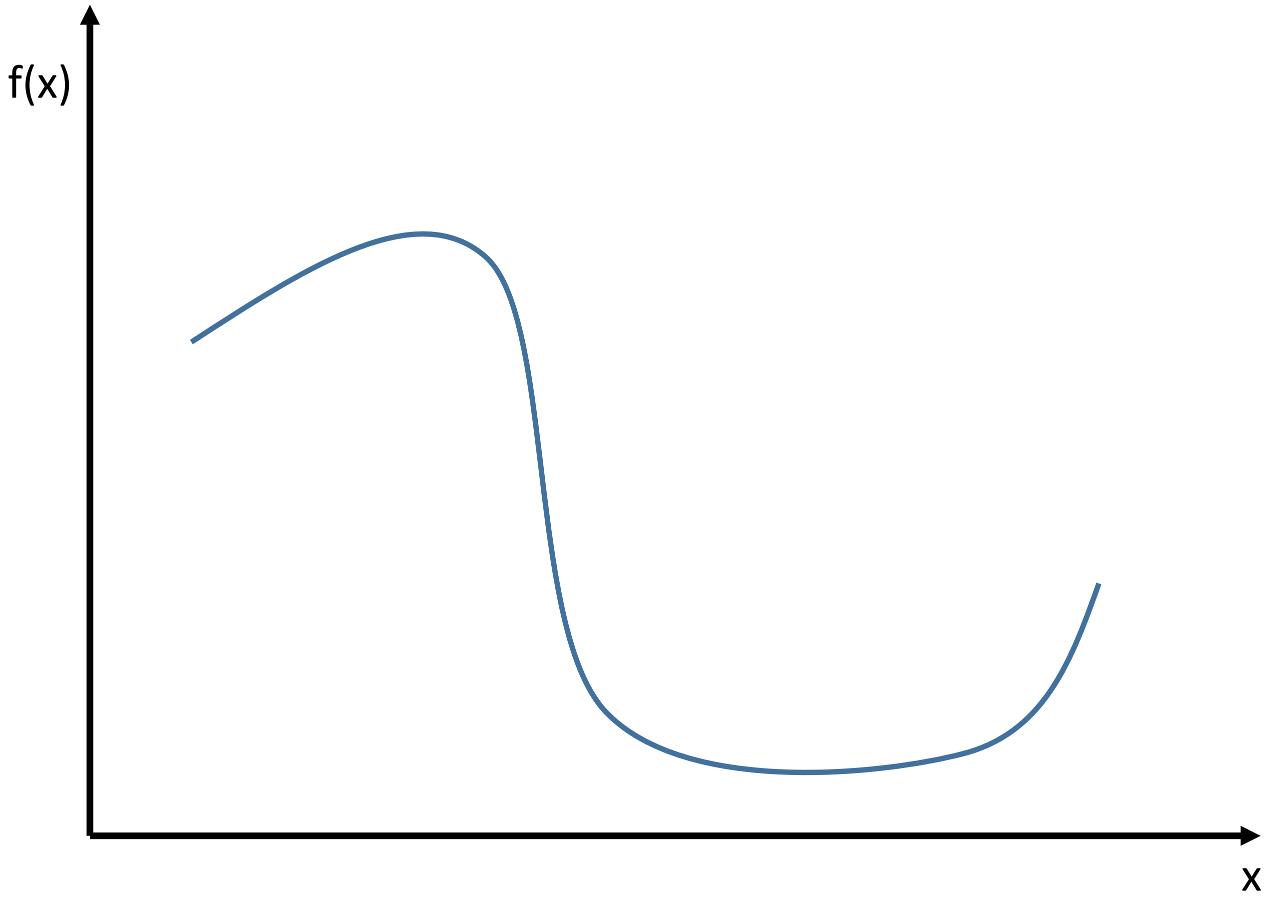


Linearization

We construct a linear approximation using function g's value and slope.

$$x_t = g(u_t, x_{t-1}) + \epsilon$$

First-order Taylor Expansion



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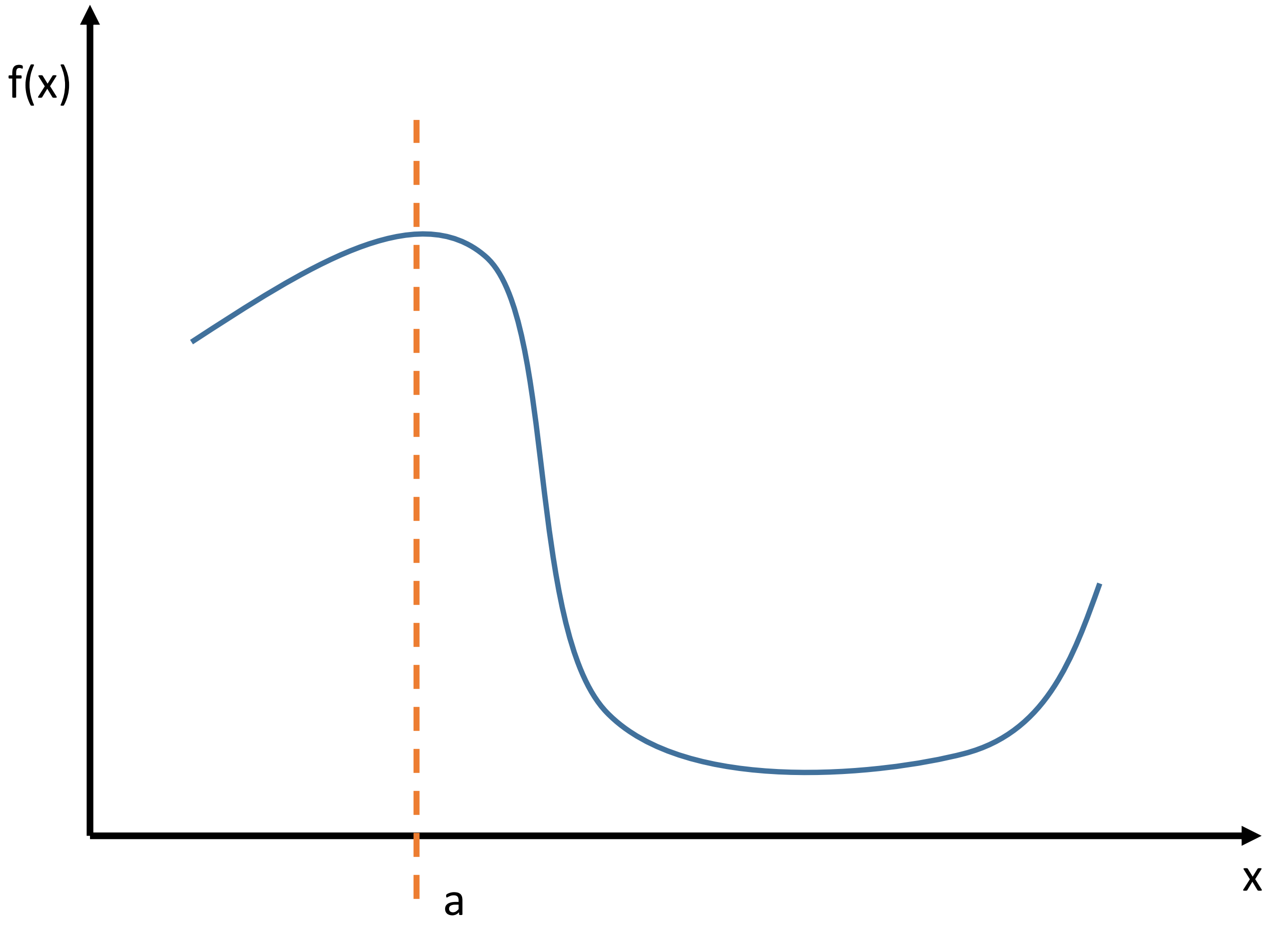
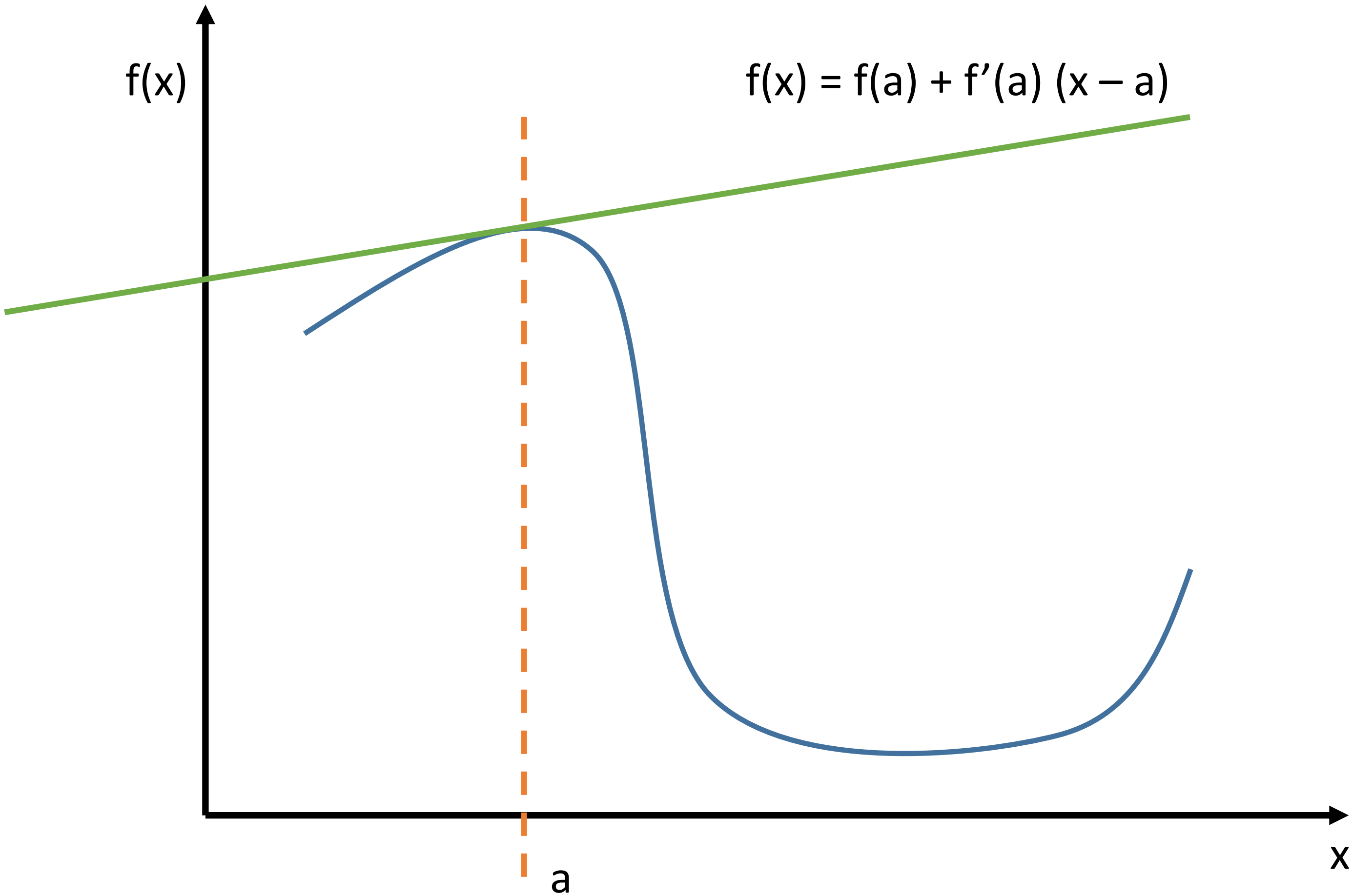


First-order Taylor Expansion



Linearization

We construct a linear approximation using function g's value and slope.

$$x_t = g(u_t, x_{t-1}) + \epsilon$$

Slope: $g'(u_t, x_{t-1}) = \dfrac{\partial g(u_t, x_{t-1})}{\partial x_{t-1}}$

Linearization

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But what about the value?

Linearization

We construct a linear approximation using function g's value and slope.

$$x_t = g(u_t, x_{t-1}) + \epsilon$$

$$\text{Slope: } g'(u_t, x_{t-1}) = \frac{\partial g(u_t, x_{t-1})}{\partial x_{t-1}}$$

But what about the value?

$$\begin{aligned} g(u_t, x_{t-1}) &\approx g(u_t, \mu_{t-1}) + g'(u_t, \mu_{t-1}) * (x_{t-1} - \mu_{t-1}) \\ &\approx g(u_t, \mu_{t-1}) + G_t(u_t, \mu_{t-1}) * (x_{t-1} - \mu_{t-1}) \end{aligned}$$

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We construct a linear approximation using function g's value and slope.

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Slope: $g'(u_t, x_{t-1}) = \dfrac{\partial g(u_t, x_{t-1})}{\partial x_{t-1}}$

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$p(x_t|u_t, x_{t-1}) =$

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We construct a linear approximation using function g's value and slope.

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$$\begin{aligned} p(x_t|u_t, x_{t-1}) =& \det(2\pi R)^{-\frac{1}{2}} \exp\{-\frac{1}{2}(x_t - g(u_t, \mu_{t-1}) - G_t(u_t, \mu_{t-1})(x_{t-1} - \mu_{t-1}))^T R^{-1} \\ & (x_t - g(u_t, \mu_{t-1}) - G_t(u_t, \mu_{t-1})(x_{t-1} - \mu_{t-1})\} \end{aligned}$$

Measurements

$$z_t = h(x_t) + \delta$$

Measurements

$$z_t = h(x_t) + \delta$$

We construct a linear approximation using function h's value and slope.

$$h(x_t) \approx h(\bar{\mu}_t) + H(\mu_t)(x_t - \bar{\mu}_t)$$

Kalman Filter

Algorithm Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$**):**

$\bar{\mu}_t = A_t \, \mu_{t-1} + B_t \, u_t$

$\bar{\Sigma}_t = A_t \, \Sigma_{t-1} \, A_t^T + R_t$

$K_t = \bar{\Sigma}_t \, C_t^T (C_t \, \bar{\Sigma}_t \, C_t^T + Q_t)^{-1}$

$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \, \bar{\mu}_t)$

$\Sigma_t = (I - K_t \, C_t) \, \bar{\Sigma}_t$

return μ_t, Σ_t

Extended Kalman Filter

Algorithm Extended_Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$**):**

$\bar{\mu}_t = g(u_t, \mu_{t-1})$

$\bar{\Sigma}_t = G_t \; \Sigma_{t-1} \; G_t^T + R_t$

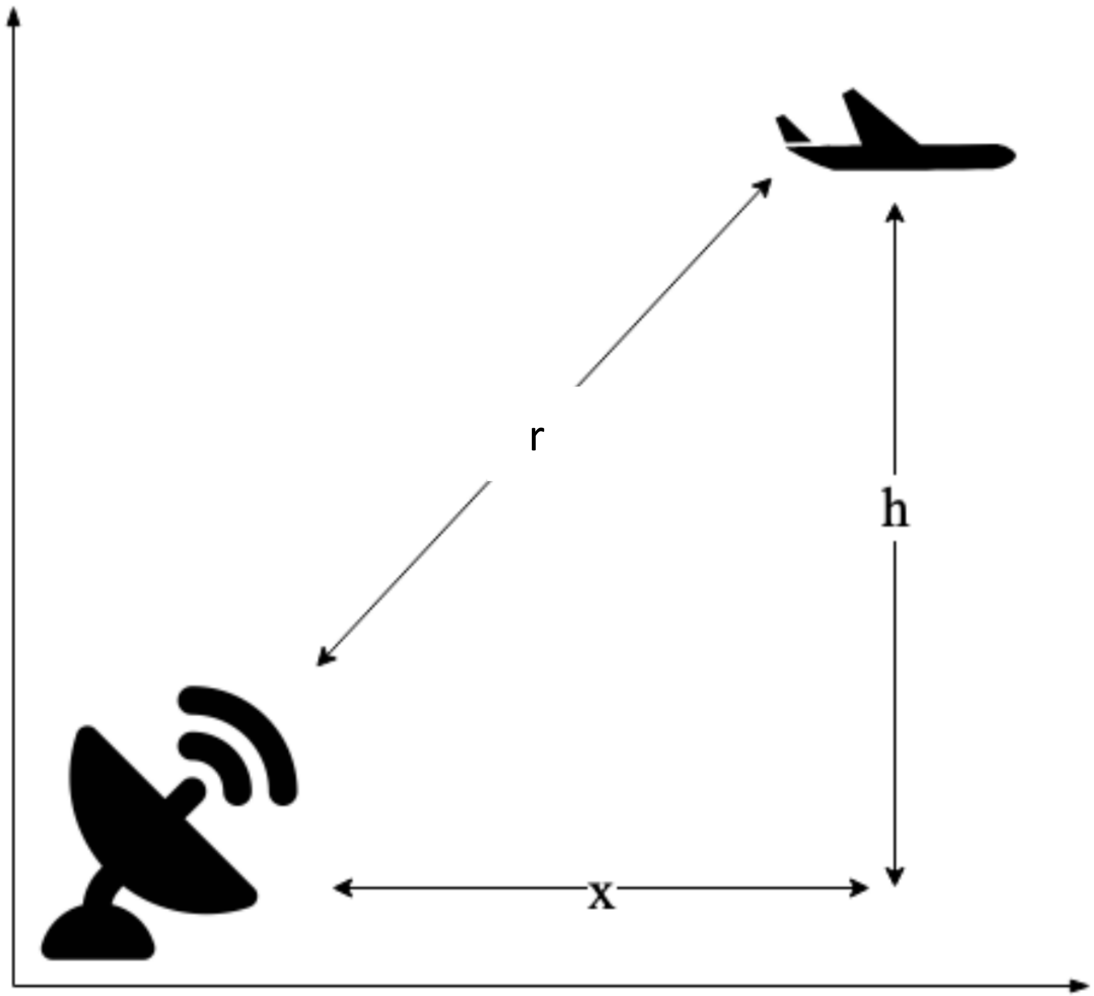
$K_t = \bar{\Sigma}_t \; H_t^T (H_t \; \bar{\Sigma}_t \; H_t^T + Q_t)^{-1}$

$\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$

$\Sigma_t = (I - K_t \; H_t) \; \bar{\Sigma}_t$

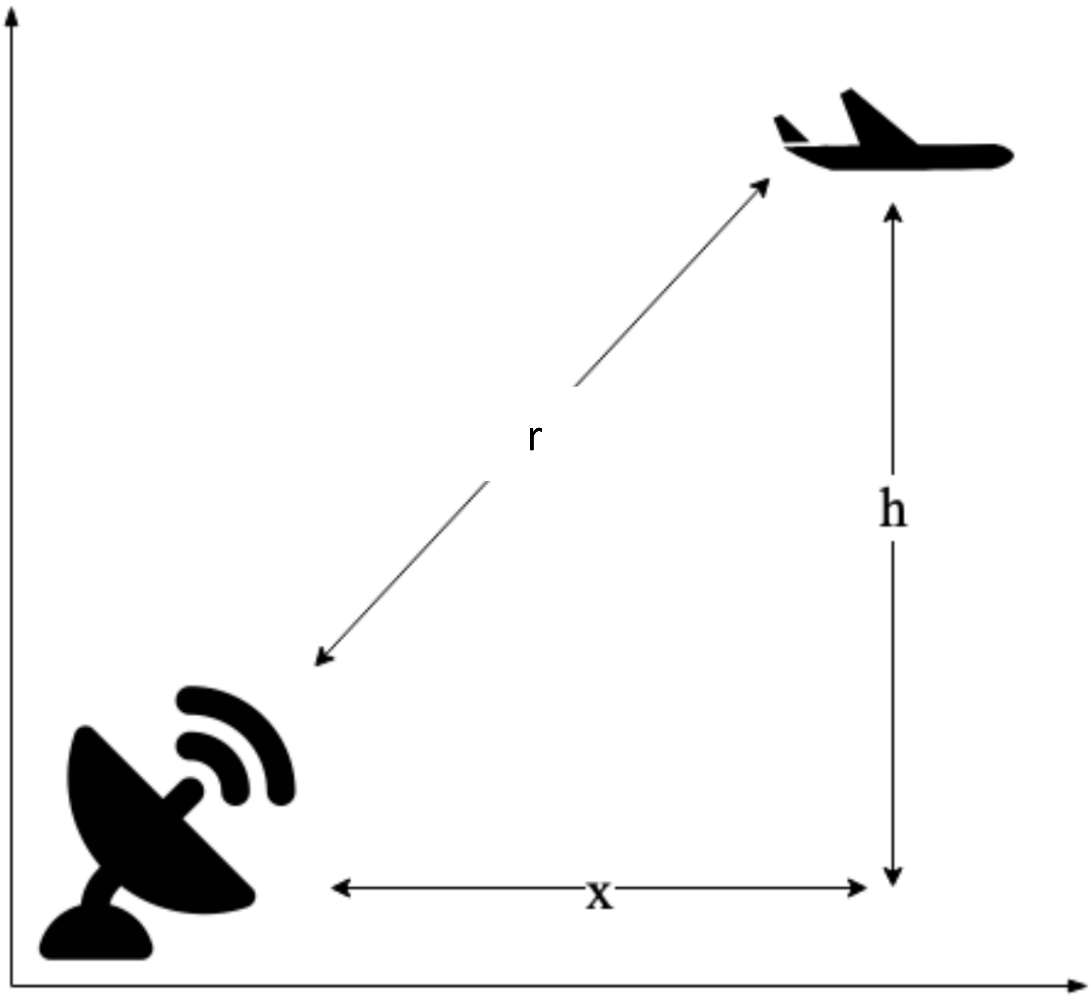
return μ_t, Σ_t

Example



$$\dot{x} = v$$

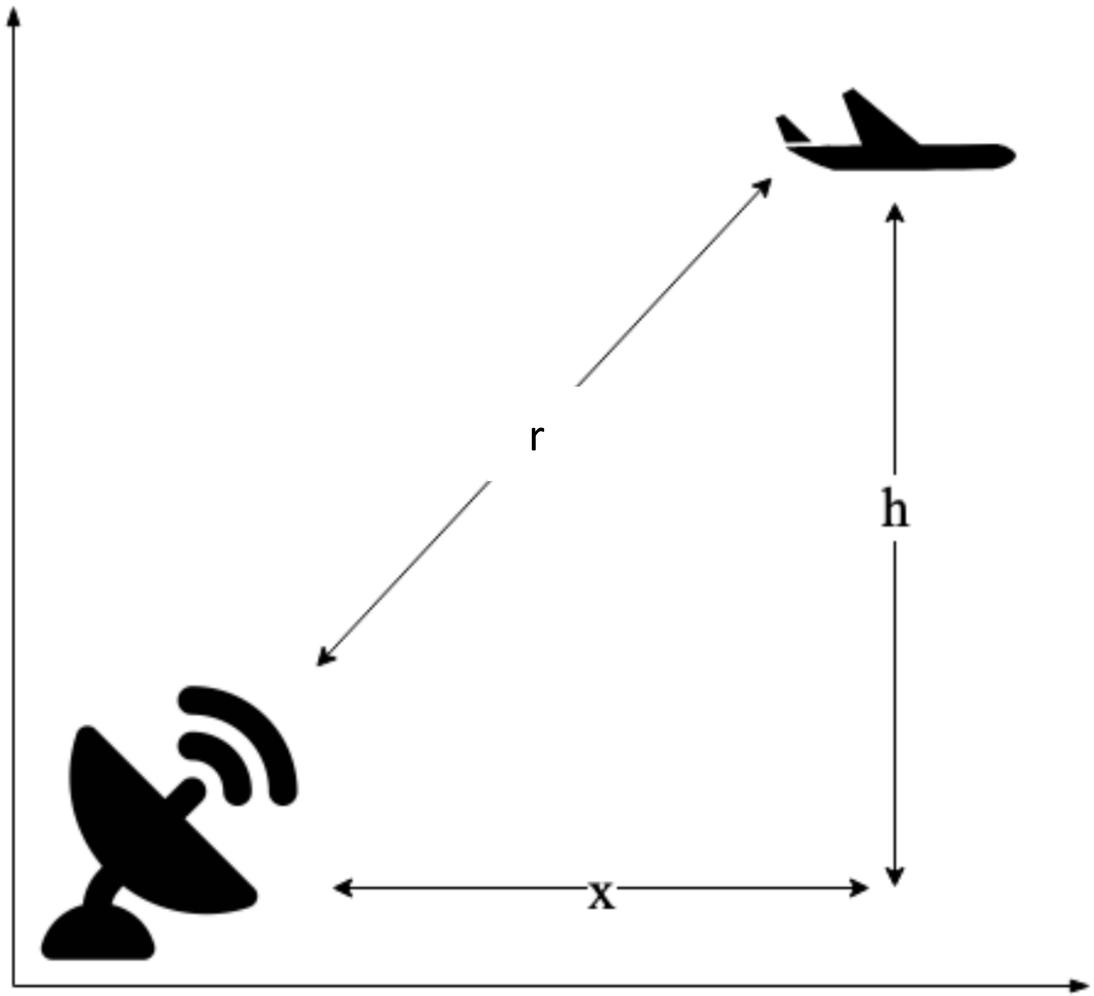
Example



$$\dot{x} = v$$

$$\dot{v} = 0$$

Example

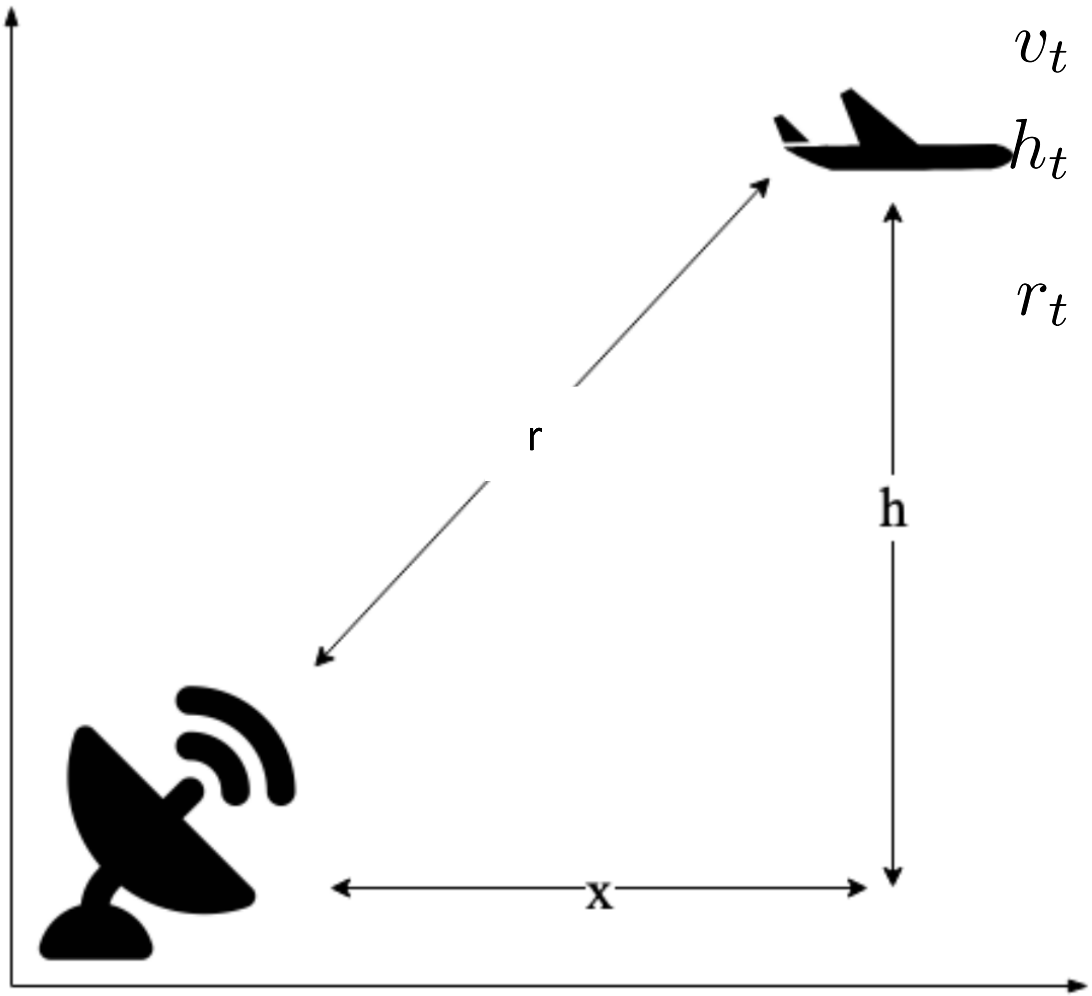


$$\dot{x} = v$$

$$\dot{v} = 0$$

$$\dot{h} = 0$$

Example



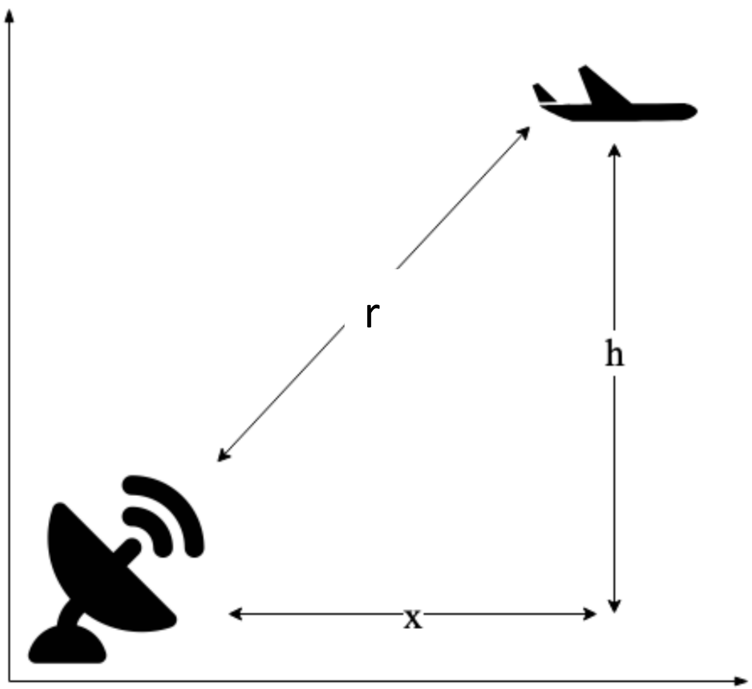
$$x_t = x_{t-1} + v_{t-1} * \Delta_t$$

$$v_t = v_{t-1}$$

$$h_t = h_{t-1}$$

$$r_t = \sqrt{x_t^2 + h_t^2}$$

Example



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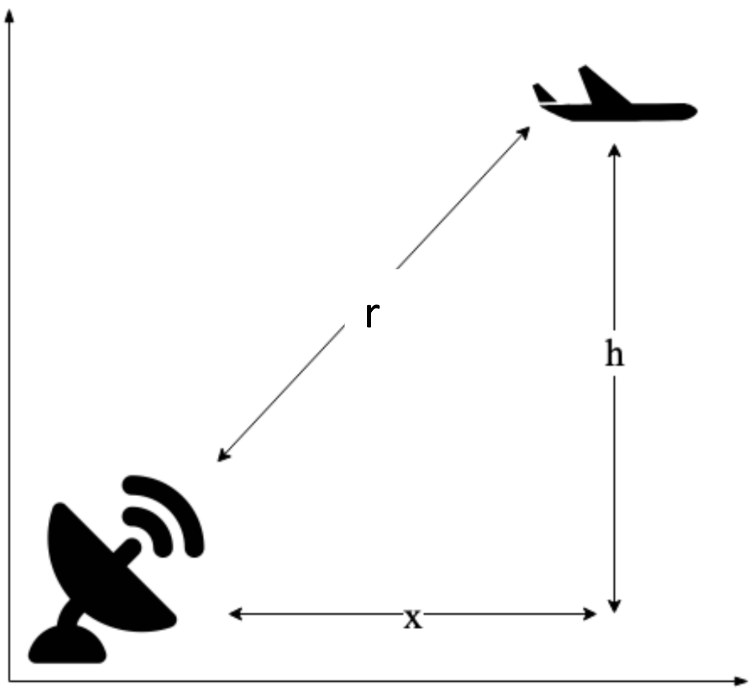
$$r_t = \sqrt{x_t^2 + h_t^2}$$

Example



$$x_t = x_{t-1} + v_{t-1} * \Delta_t$$

$$v_t = v_{t-1}$$

$$h_t = h_{t-1}$$

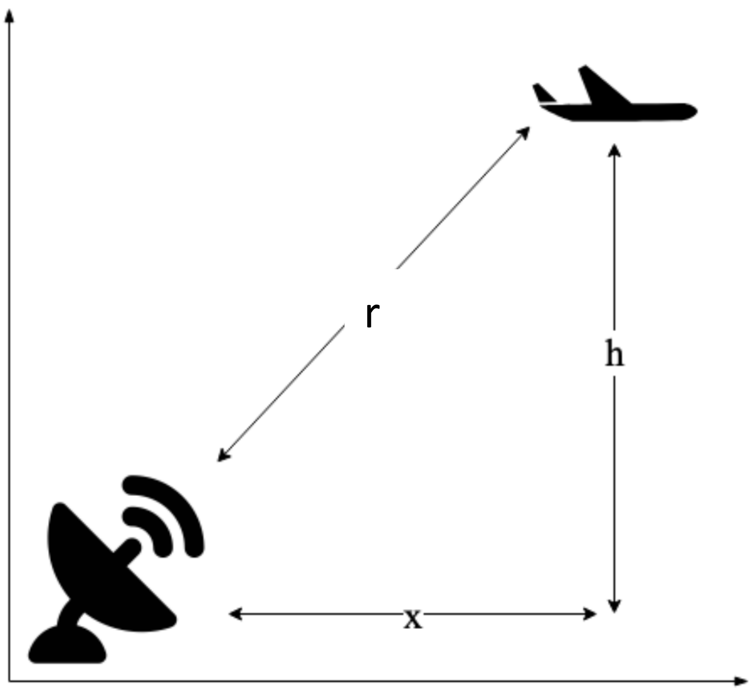
$$r_t = \sqrt{x_t^2 + h_t^2}$$

$$x_{1,t} = x_t$$

$$x_{2,t} = v_t$$

$$x_{3,t} = h_t$$

Example



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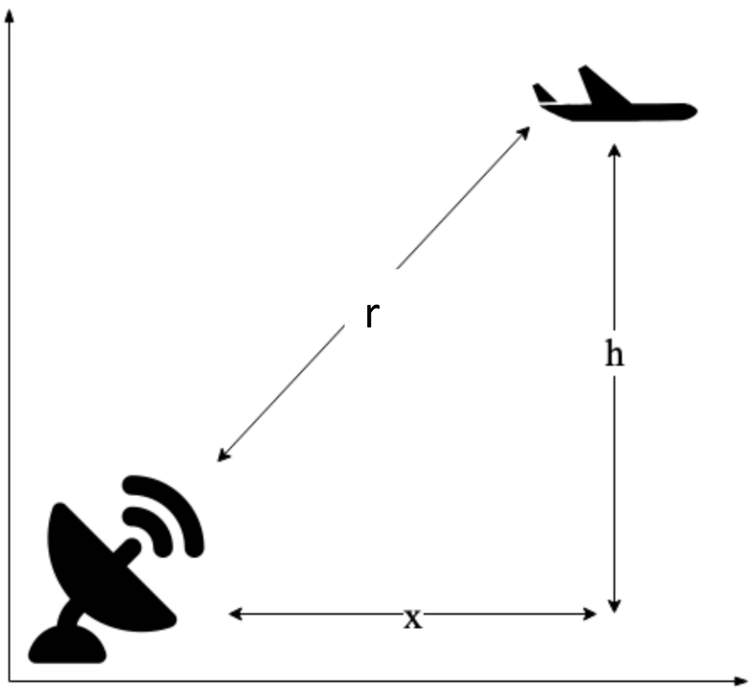
$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \cdot \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \\ x_{3,t-1} \end{bmatrix}$$

Example



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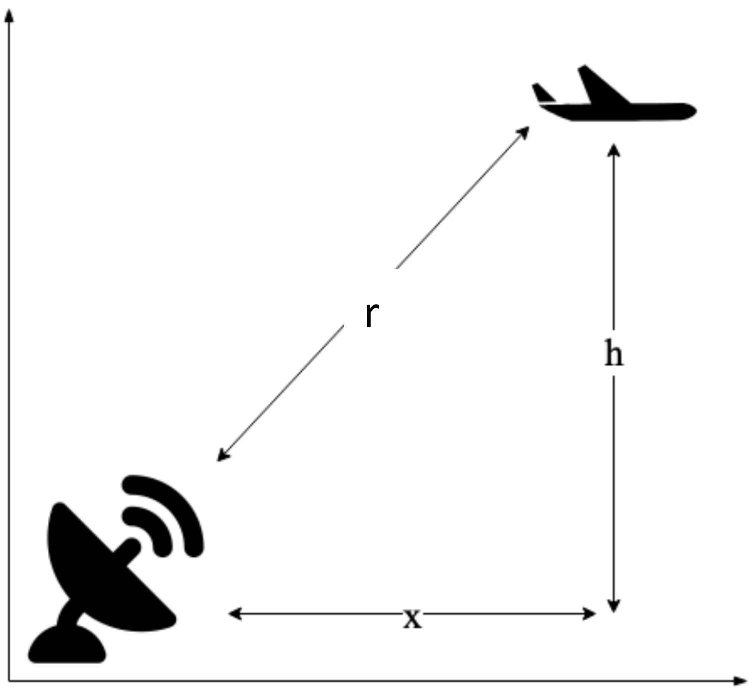
$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} = \begin{bmatrix} 1 & \Delta_t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \\ x_{3,t-1} \end{bmatrix}$$

Example



$$x_t = x_{t-1} + v_{t-1} * \Delta_t$$

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Are the dynamics linear?

Example

$$g(x_{t-1}, u_t) \approx g(u_t, \mu_{t-1}) + G_t(u_t, \mu_{t-1}) * (x_{t-1} - \mu_{t-1})$$

Example

$$
\begin{aligned}
g(x_{t-1}, u_t) &\approx g(u_t, \mu_{t-1}) + G_t(u_t, \mu_{t-1}) * (x_{t-1} - \mu_{t-1}) \\
&\approx \begin{bmatrix} \mu_{1,t-1} + \Delta_t \mu_{2,t-1} \\ \mu_{2,t-1} \\ \mu_{3,t-1} \end{bmatrix} +
\end{aligned}
$$

Example

$$g(x_{t-1}, u_t) \approx g(u_t, \mu_{t-1}) + G_t(u_t, \mu_{t-1}) * (x_{t-1} - \mu_{t-1})$$

$$\approx \begin{bmatrix} \mu_{1,t-1} + \Delta_t \mu_{2,t-1} \\ \mu_{2,t-1} \\ \mu_{3,t-1} \end{bmatrix} + \begin{bmatrix} 1 & \Delta_t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,t-1} - \mu_{t,t-1} \\ x_{2,t-1} - \mu_{2,t-1} \\ x_{3,t-1} - \mu_{3,t-1} \end{bmatrix}$$

Example

$$g(x_{t-1}, u_t) \approx g(u_t, \mu_{t-1}) + G_t(u_t, \mu_{t-1}) * (x_{t-1} - \mu_{t-1})$$

$$\approx \begin{bmatrix} \mu_{1,t-1} + \Delta_t \mu_{2,t-1} \\ \mu_{2,t-1} \\ \mu_{3,t-1} \end{bmatrix} + \begin{bmatrix} 1 & \Delta_t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,t-1} - \mu_{t,t-1} \\ x_{2,t-1} - \mu_{2,t-1} \\ x_{3,t-1} - \mu_{3,t-1} \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & \Delta_t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_{t,t-1} \\ \mu_{2,t-1} \\ \mu_{3,t-1} \end{bmatrix} + \begin{bmatrix} 1 & \Delta_t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,t-1} - \mu_{t,t-1} \\ x_{2,t-1} - \mu_{2,t-1} \\ x_{3,t-1} - \mu_{3,t-1} \end{bmatrix}$$

Extended Kalman Filter

Algorithm Extended_Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$**):**

$\bar{\mu}_t = g(u_t, \mu_{t-1})$

$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$

$K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$

$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$

$\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$

return μ_t, Σ_t

Kalman Filter

Algorithm Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$**):**

$\bar{\mu}_t = A_t \, \mu_{t-1} + B_t \, u_t$

$\bar{\Sigma}_t = A_t \, \Sigma_{t-1} \, A_t^T + R_t$

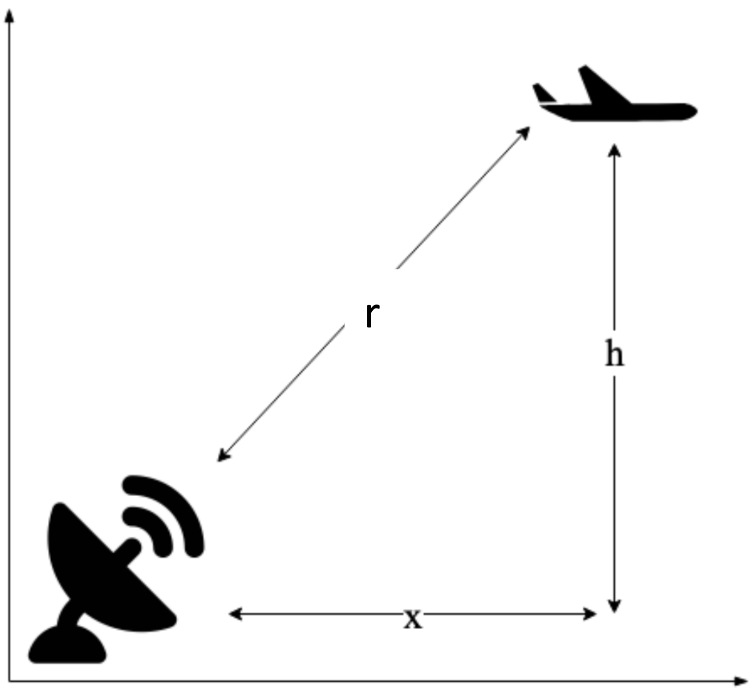
$K_t = \bar{\Sigma}_t \, C_t^T (C_t \, \bar{\Sigma}_t \, C_t^T + Q_t)^{-1}$

$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \, \bar{\mu}_t)$

$\Sigma_t = (I - K_t \, C_t) \, \bar{\Sigma}_t$

return μ_t, Σ_t

Example



$$x_t = x_{t-1} + v_{t-1} * \Delta_t$$

$$v_t = v_{t-1}$$

$$h_t = h_{t-1}$$

$$r_t = \sqrt{x_t^2 + h_t^2}$$

$$x_{1,t} = x_t$$

$$x_{2,t} = v_t$$

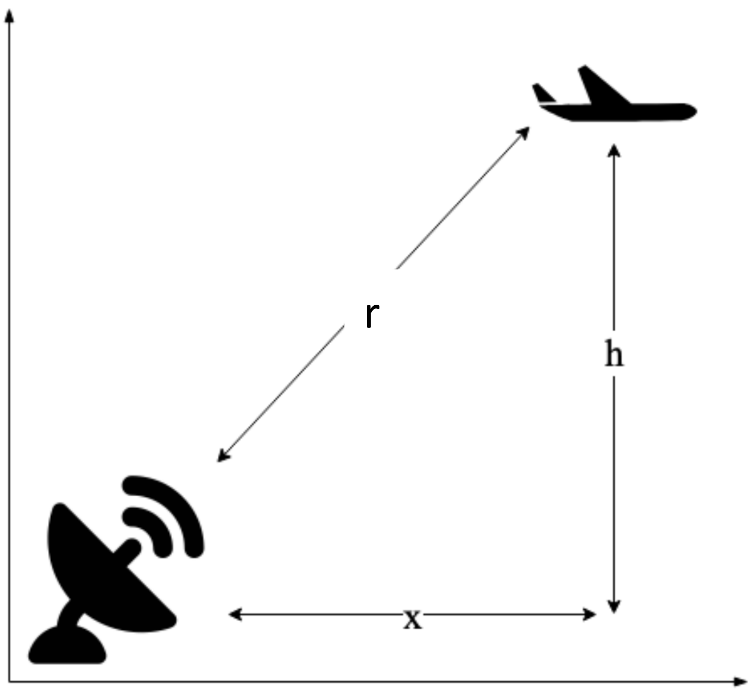
$$x_{3,t} = h_t$$

$$z_t = h(x_t)$$

$$= \sqrt{x_{1,t}^2 + x_{3,t}^2} + \delta$$

Are the measurements linear?

Example



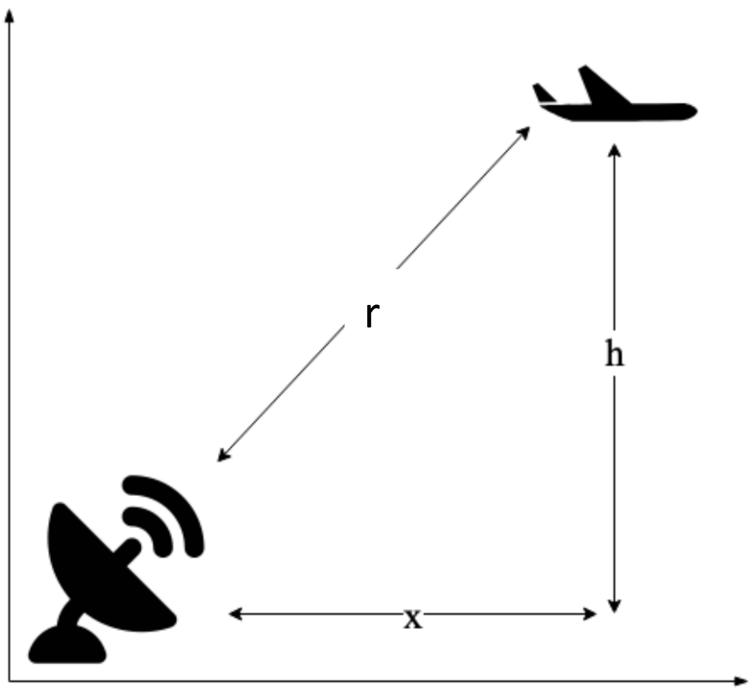
$$x_{t+1} = x_t + v_t * \Delta_t$$
$$v_{t+1} = v_t$$
$$h_{t+1} = h_t$$
$$r_t = \sqrt{x_t^2 + h_t^2}$$

$$x_{1,t} = x_t$$
$$x_{2,t} = v_t$$
$$x_{3,t} = h_t$$

$$z_t = h(x_t) + \delta$$
$$= \sqrt{x_{1,t}^2 + x_{3,t}^2} + \delta$$

We will linearize at the mean!

Example



$$x_{t+1} = x_t + v_t * \Delta_t$$

$$v_{t+1} = v_t$$

$$h_{t+1} = h_t$$

$$r_t = \sqrt{x_t^2 + h_t^2}$$

$$x_{1,t} = x_t$$

$$x_{2,t} = v_t$$

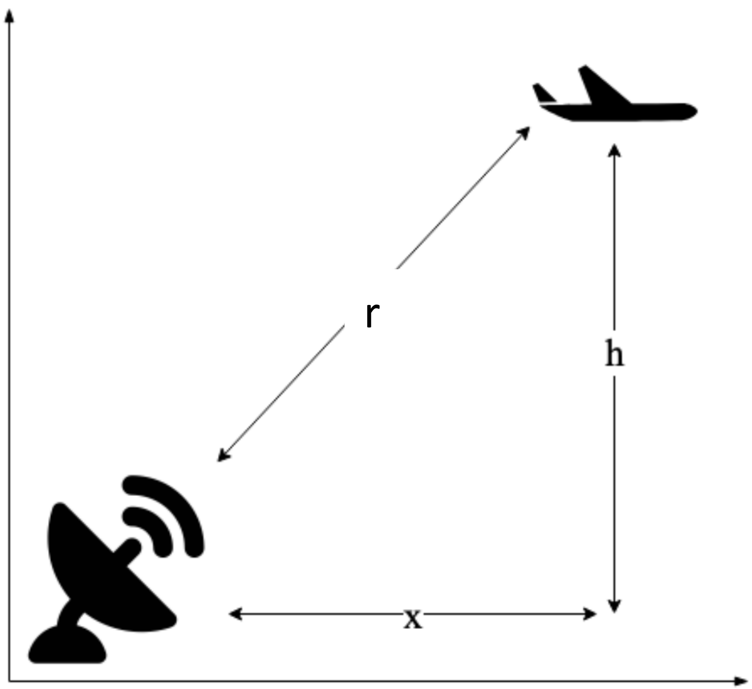
$$x_{3,t} = h_t$$

$$z_t = h(x_t) + \delta$$

$$= \sqrt{x_{1,t}^2 + x_{3,t}^2} + \delta$$

$$h(x_t) = h(\mu_t) + H_t(\mu_t)(z_t - \mu_t)$$

Example



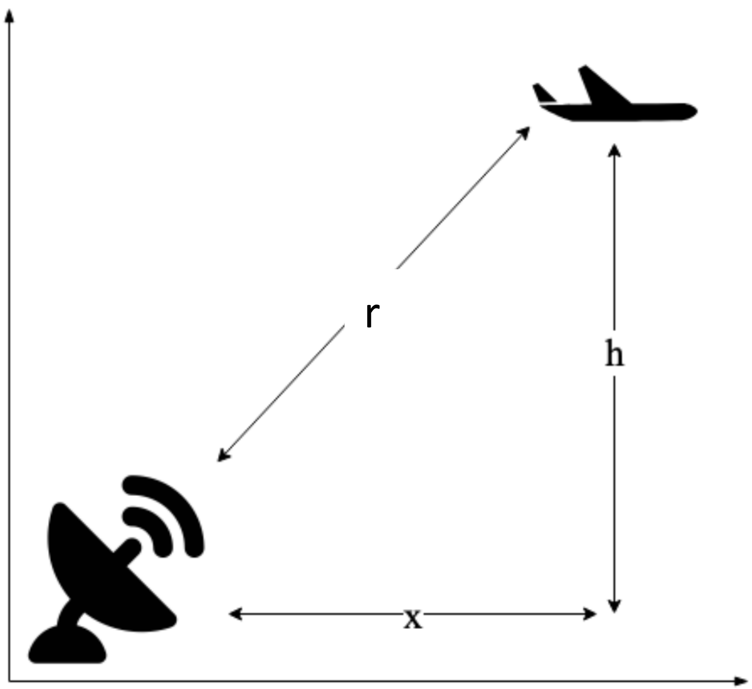
$$x_{t+1} = x_t + v_t * \Delta_t$$
$$v_{t+1} = v_t$$
$$h_{t+1} = h_t$$
$$r_t = \sqrt{x_t^2 + h_t^2}$$

$$h(x_t) = h(\mu_t) + H_t(\mu_t)(x_t - \mu_t)$$

$$z_t = h(x_t) + \delta$$
$$= \sqrt{x_{1,t}^2 + x_{3,t}^2} + \delta$$

$$H_t = \begin{bmatrix} \frac{\partial h}{\partial x_{t,1}} & \frac{\partial h}{\partial x_{t,2}} & \frac{\partial h}{\partial x_{t,3}} \end{bmatrix}$$

Example



$$x_{t+1} = x_t + v_t * \Delta_t$$

$$v_{t+1} = v_t$$

$$h_{t+1} = h_t$$

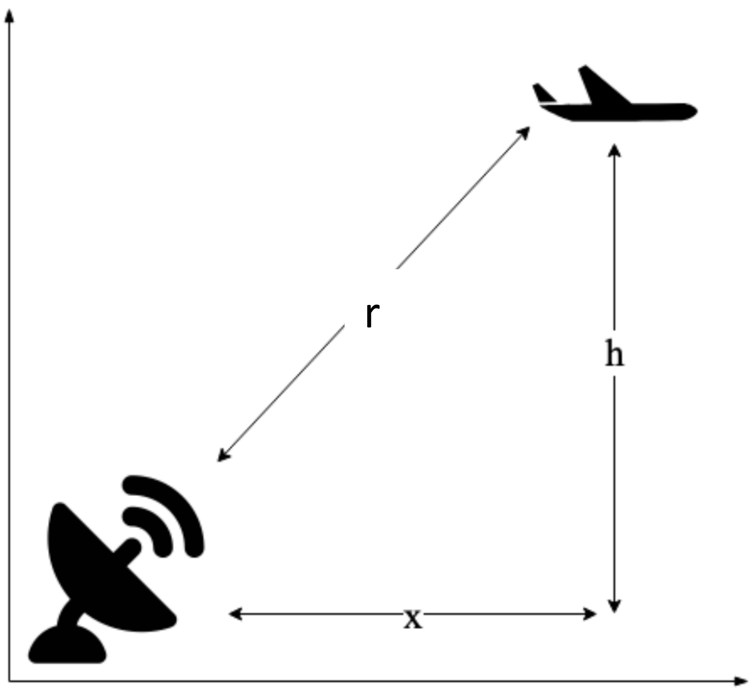
$$r_t = \sqrt{x_t^2 + h_t^2}$$

$$h(x_t) = h(\mu_t) + H_t(\mu_t)(x_t - \mu_t)$$

$$z_t = h(x_t) + \delta$$
$$= \sqrt{x_{1,t}^2 + x_{3,t}^2} + \delta$$

$$H_t = \begin{bmatrix} \frac{\partial h}{\partial x_{t,1}} & \frac{\partial h}{\partial x_{t,2}} & \frac{\partial h}{\partial x_{t,3}} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{2x_{t,1}}{2\sqrt{x_{t,1}^2 + x_{t,3}^2}} & 0 & \frac{2x_{t,3}}{2\sqrt{x_{t,1}^2 + x_{t,3}^2}} \end{bmatrix}$$

Example



$$x_{t+1} = x_t + v_t * \Delta_t$$

$$v_{t+1} = v_t$$

$$h_{t+1} = h_t$$

$$r_t = \sqrt{x_t^2 + h_t^2}$$

$$z_t = h(x_t) + \delta$$

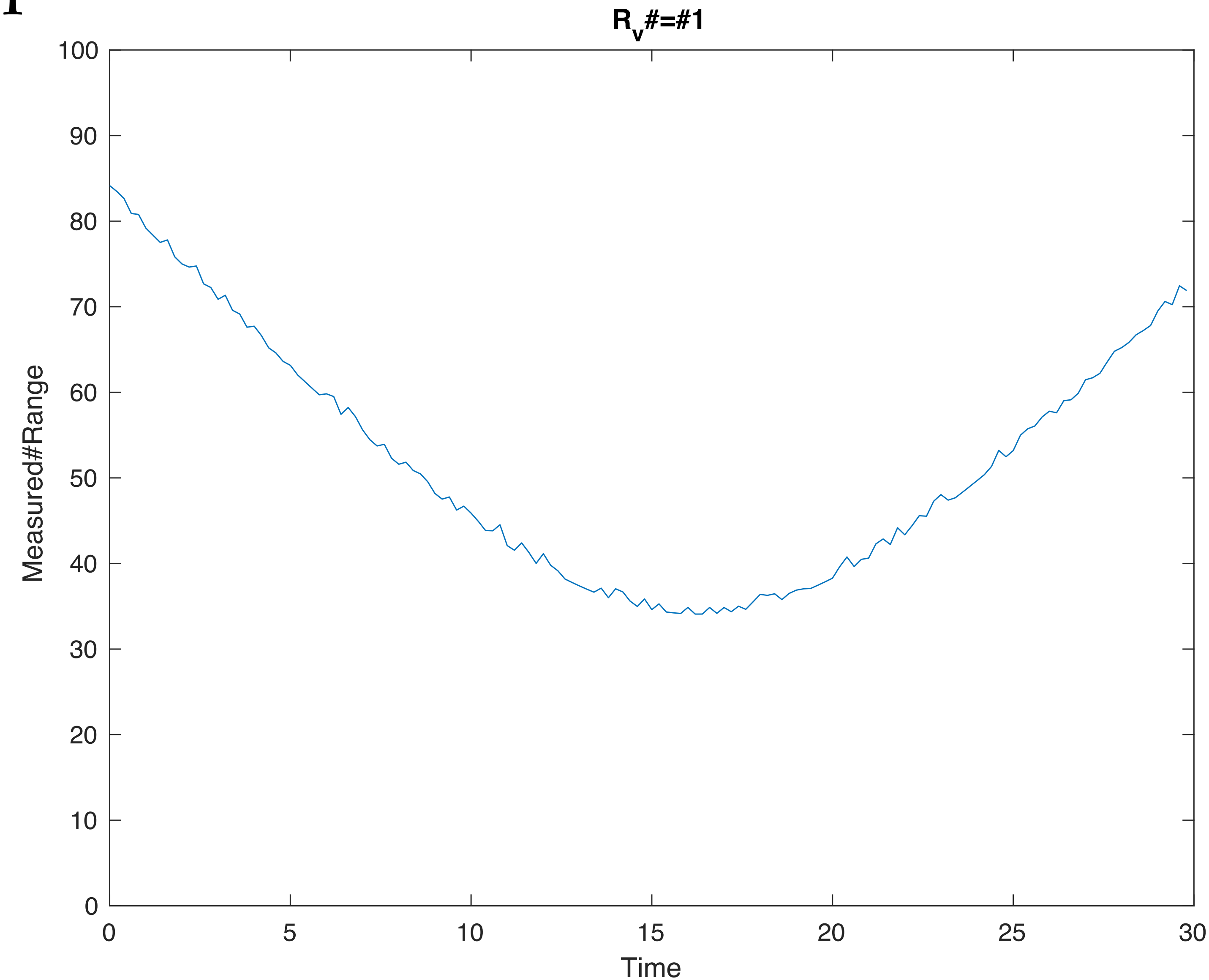
$$= \sqrt{x_{1,t}^2 + x_{3,t}^2} + \delta$$

$$h(x_t) = h(\mu_t) + H_t(\mu_t)(x_t - \mu_t)$$

$$H_t = \left[\frac{\partial h}{\partial x_{t,1}} \quad \frac{\partial h}{\partial x_{t,2}} \quad \frac{\partial h}{\partial x_{t,3}} \right]$$

$$= \left[\frac{2x_{t,1}}{2\sqrt{x_{t,1}^2 + x_{t,3}^2}} \quad 0 \quad \frac{2x_{t,3}}{2\sqrt{x_{t,1}^2 + x_{t,3}^2}} \right]$$

Example



Measured range *r*

Example