

Mathematical Programming

CSCI 545 Introduction to Robotics
Instructor: Stefanos Nikolaidis

Resources : *16-811: Math Fundamentals for Robotics* by Michael
Erdmann, Carnegie Mellon University

Optimization Problems

- How can a robot reach a specific object while avoiding obstacles?
- How can a robot go to its destination while minimizing energy consumption?

Basic Problem

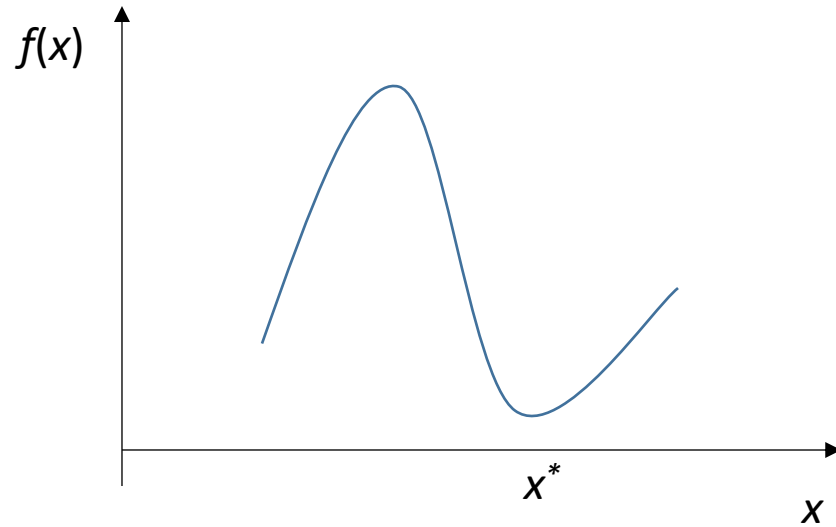
- Given $f : \mathbb{R}^n \rightarrow \mathbb{R}$ find $\min f$

Basic Problem

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- Let's start with $n = 1$

$$f : \mathbb{R} \rightarrow \mathbb{R}$$



Basic Problem

- Find $\min f : \mathbb{R} \rightarrow \mathbb{R}$

- We compute the critical set of f :

$$C_f = \{x \mid f'(x) = 0\}$$

- We identify minima by finding where $f''(x) \geq 0$

Basic Problem

- What about the n dimensional case?

Basic Problem

- What about the n dimensional case?
- Sufficient conditions for a relative (local) case:
- We let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be C^2
- We let x^* be a relative minimum of f :
 - Then: i) $\nabla f(x^*) = 0$
ii) $\nabla^2 f(x^*) \succ 0$ (positive definite)

Basic Problem

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix},$$

Example

$$c \in \mathbb{R}$$

$$b \in \mathbb{R}^n$$

$$A = (a_{ij}) \in \mathbb{R}^{n \times n}, a_{ij} = a_{ji}, A \succ 0$$

$$f(x) = c + b^T x + 0.5x^T A x$$

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$$\nabla f(x)?$$

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$$\nabla f(x) = b + Ax$$

Example (n=1)

$$a > 0$$

$$f(x) = c + bx + \frac{1}{2}ax^2$$

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Example (n=1)

$$a > 0$$

$$f(x) = c + bx + \frac{1}{2}ax^2$$

$$f'(x) = b + ax$$

$$f''(x) = a > 0$$

$$f'(x^*) = 0$$

$$x^* = -\frac{b}{a}$$

Example (n=2)

- Say $x = (x_1, x_2)$:

$$f(x) = c + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

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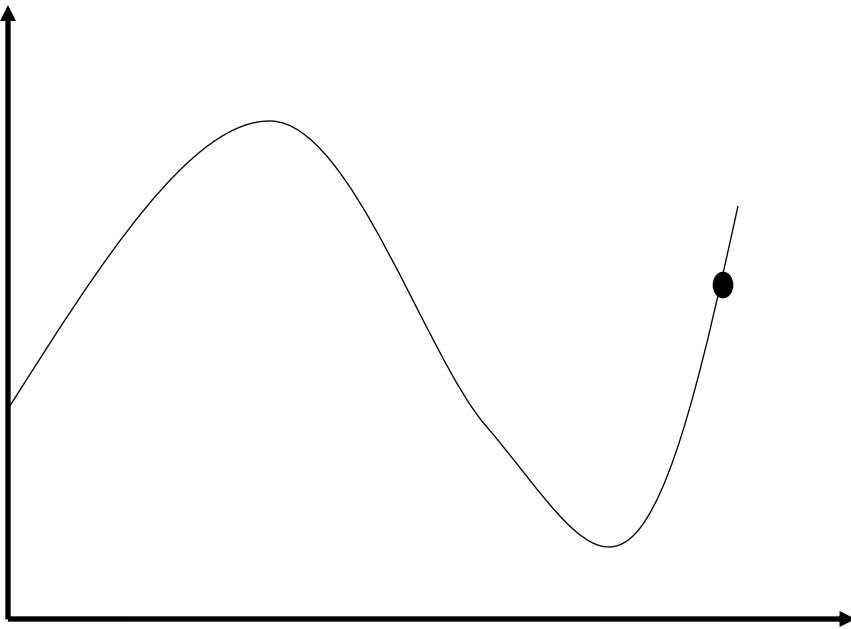
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There is a local minimum that is the solution to the equation: $Ax^* = -b$

Since it is the only one, this is a global minimum.

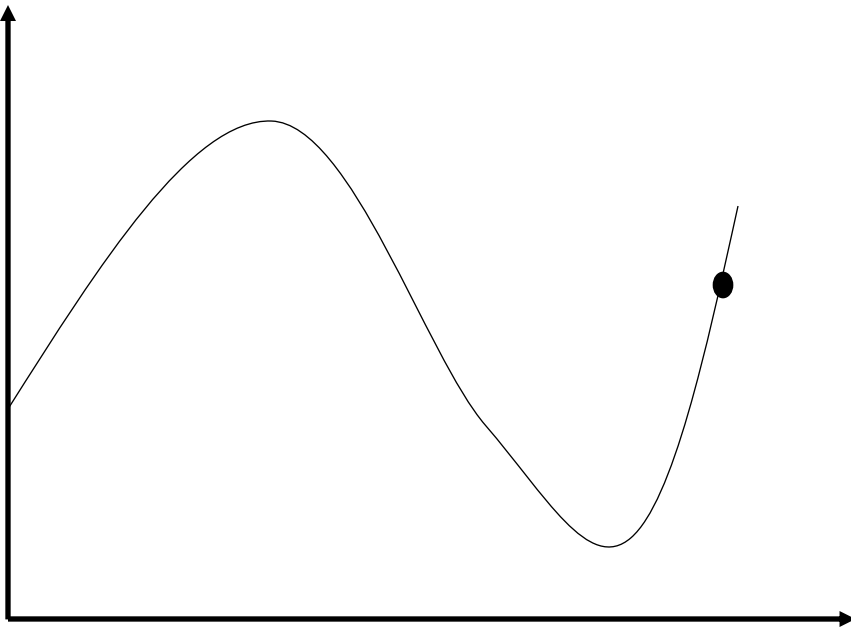
Gradient Descent

- Given $f : \mathbb{R} \rightarrow \mathbb{R}$



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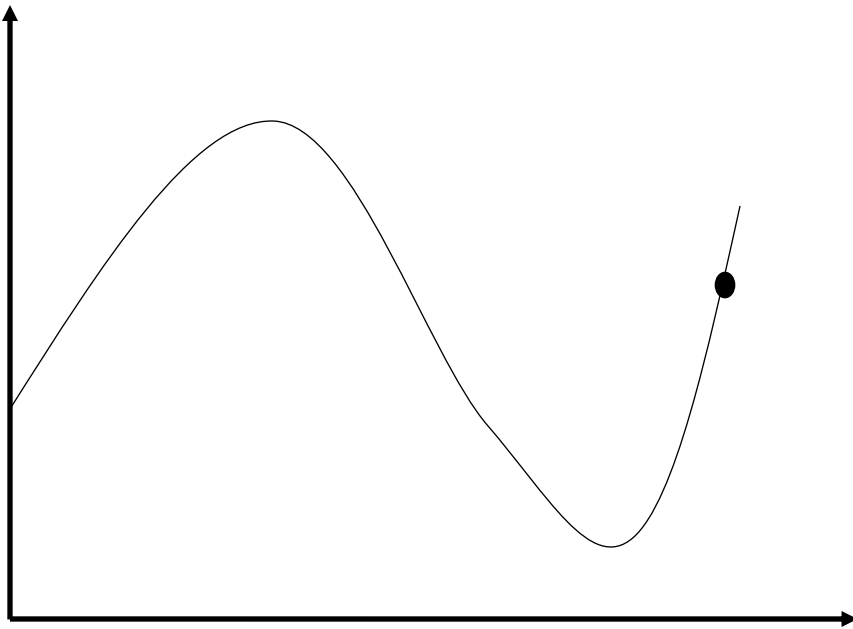


Rule for finding minimum:

- if $f'(x) < 0$ move right
- if $f'(x) > 0$ move left
- if $f'(x) = 0$ stop

Gradient Descent

- Given $f : \mathbb{R} \rightarrow \mathbb{R}$



In higher dimensions, we compute $\nabla_x f$

Gradient Descent

- Given $f : \mathbb{R} \rightarrow \mathbb{R}$

$$x_{t+1} = x_t + a f'(x)$$

- General case $f : \mathbb{R}^n \rightarrow \mathbb{R}$

$$x_{t+1} = x_t + a \nabla_x f(x)$$

Value of a

- A high learning rate covers more ground at each step, but we risk overshooting the minimum.
- A very low learning rate is more precise but calculating the gradient is time-consuming, so it can take a long time.

Constrained Optimization

- It is the process of optimizing an objective function in the presence of constraints.

Example Problem

- Let Fire 1 need 1000 units of water, Fire 2 need 200 units of water, Fire 3 need 3000 units of water. Aircraft A can deliver 1 unit of water per unit time. Aircraft B can deliver 2 units of water per unit time.
- Goal: extinguish all the fires in minimum time.

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- Formulation: Let t_{A1} , t_{A2} , t_{A3} , the times vehicle A devotes to fire 1, 2, 3.
- Let t_{B1} , t_{B2} , t_{B3} in a similar way, the times vehicle B devotes to fire 1, 2, 3.

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- min T , Constraints?

Example Problem

- Let Fire 1 need 1000 units of water, Fire 2 need 200 units of water, Fire 3 3000 units of water. Aircraft A can deliver 1 unit of water per unit time. Aircraft B can deliver 2 units of water per unit time.
- Goal: min T
- Constraints:
 1. total units to put out each fire
 2. total time should be T

Example Problem

- Let Fire 1 need 1000 units of water, Fire 2 need 2000 units of water, Fire 3 need 3000 units of water. Aircraft A can deliver 1 unit of water per unit time. Aircraft B can deliver 2 units of water per unit time.
- Goal: min T
- Constraints:
 1. $t_{A1} + 2t_{B1} = 1000$, $t_{A2} + 2t_{B2} = 2000$, $t_{A3} + 2t_{B3} = 3000$
 2. total time should be T

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 3. Other constraints?

Example Problem

- Let Fire 1 need 1000 units of water, Fire 2 need 2000 units of water, Fire 3 need 3000 units of water. Aircraft A can deliver 1 unit of water per unit time. Aircraft B can deliver 2 units of water per unit time.
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 2. $t_{A1} + t_{A2} + t_{A3} \leq T$, $t_{B1} + t_{B2} + t_{B3} \leq T$
 3. $t_{Ai}, t_{Bi}, T \geq 0$

Constrained Optimization

- Many problems in engineering can be defined with:
 - a set of constraints defining all candidate (“feasible”) solutions
 $g(x) \leq 0$
 - a cost function defining the “quality” of a solution, $f(x)$

Example Problem 2

$$\text{minimize } \sum_{t=0}^2 u(t)$$

subject to

$$x(t+1) = 2x(t) + u(t)$$

$$x(0) = 0$$

$$x(3) = 10$$

$$u(t) \geq 0$$

Example Problem 2

$$\text{minimize } \sum_{t=0}^2 u(t)$$

subject to

$$x(1) = 2x(0) + u(0)$$

$$x(2) = 2x(1) + u(1)$$

$$x(3) = 2x(2) + u(2)$$

$$u(t) \geq 0$$

$$x(0) = 0$$

$$x(3) = 10$$

Intuition: the earlier you provide the input the better.

Solution: $u(0) = 2.5$

Constrained Optimization

- Many problems in engineering can be defined with:
 - a set of constraints defining all candidate (“feasible”) solutions
 $g(x) \leq 0$
 - a cost function defining the “quality” of a solution, $f(x)$
- If f and g are affine function of x , these problems are called linear programs.

Linear Program Standard Form

$$\min c_1x_1 + c_2x_2 + \dots + c_nx_n = z$$

s.t.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Linear Program Standard Form

$$\min c^T x = z$$

s.t.

$$Ax = b$$

$$x \geq 0$$

- Simplex Algorithm

Intuition Behind Simplex Algorithm

$$\max z = x_1 + 2x_2$$

s.t.

$$x_1 \leq 3$$

$$x_1 + x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

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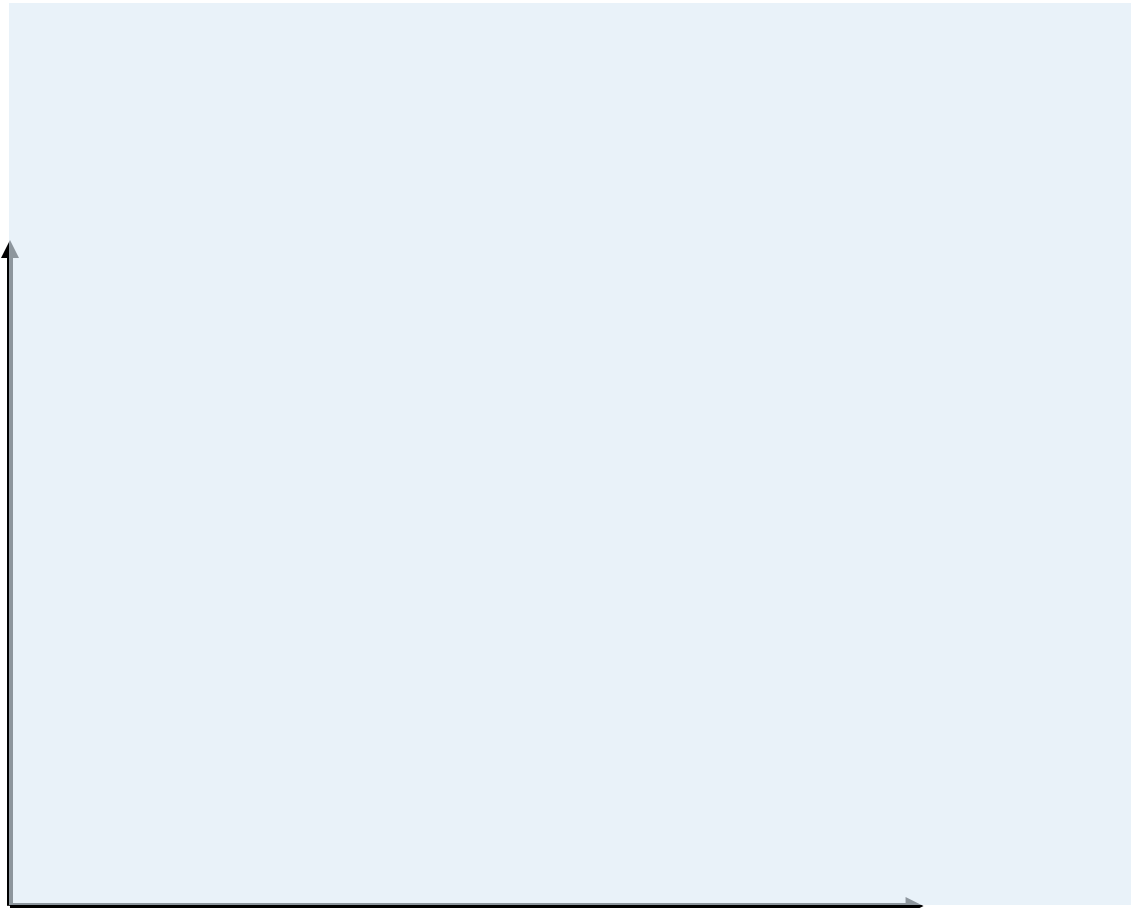
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x_2



x_1

Intuition Behind Simplex Algorithm

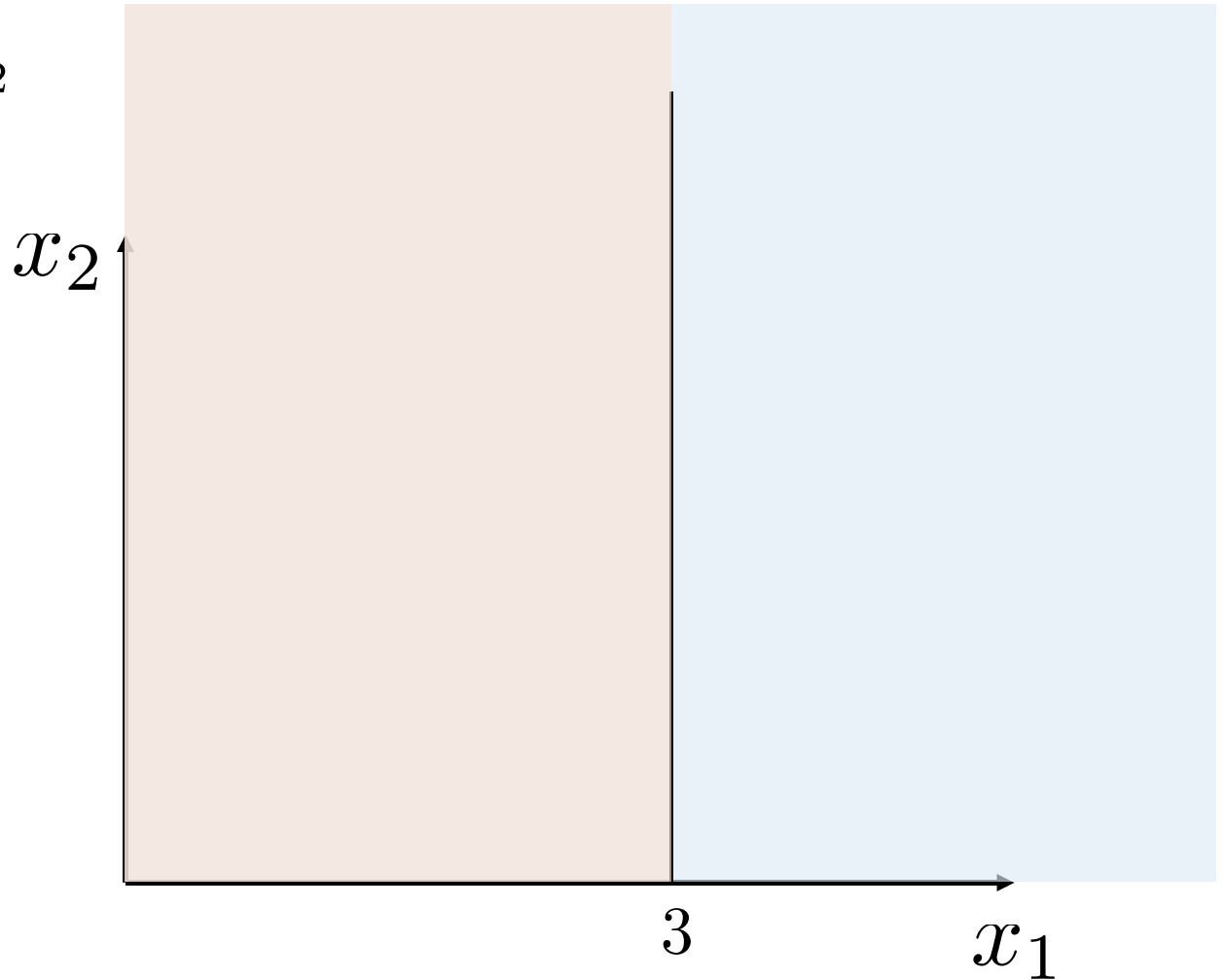
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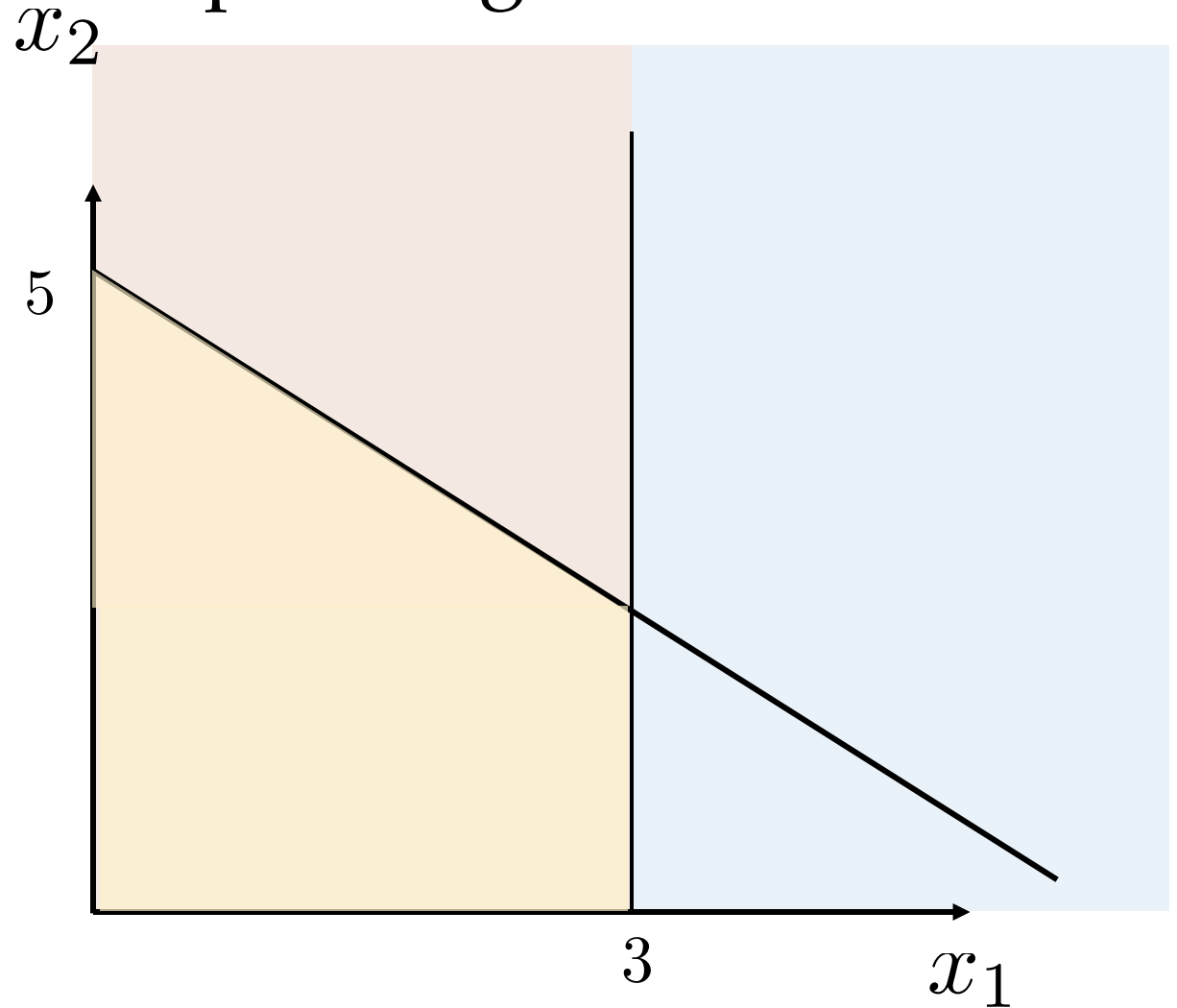
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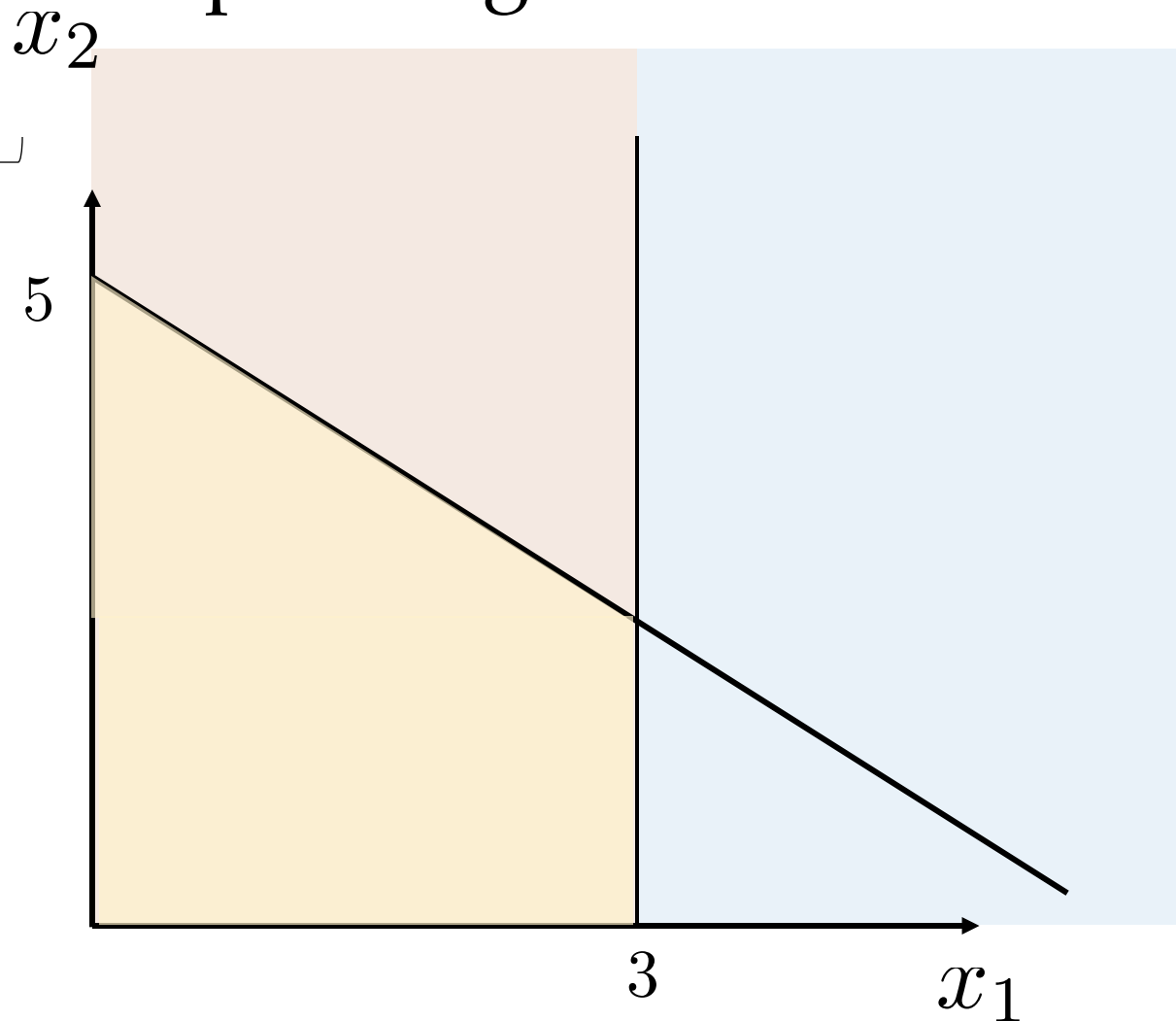
Intuition Behind Simplex Algorithm

→ $\max z = x_1 + 2x_2$
s.t. $f(x_1, x_2)$

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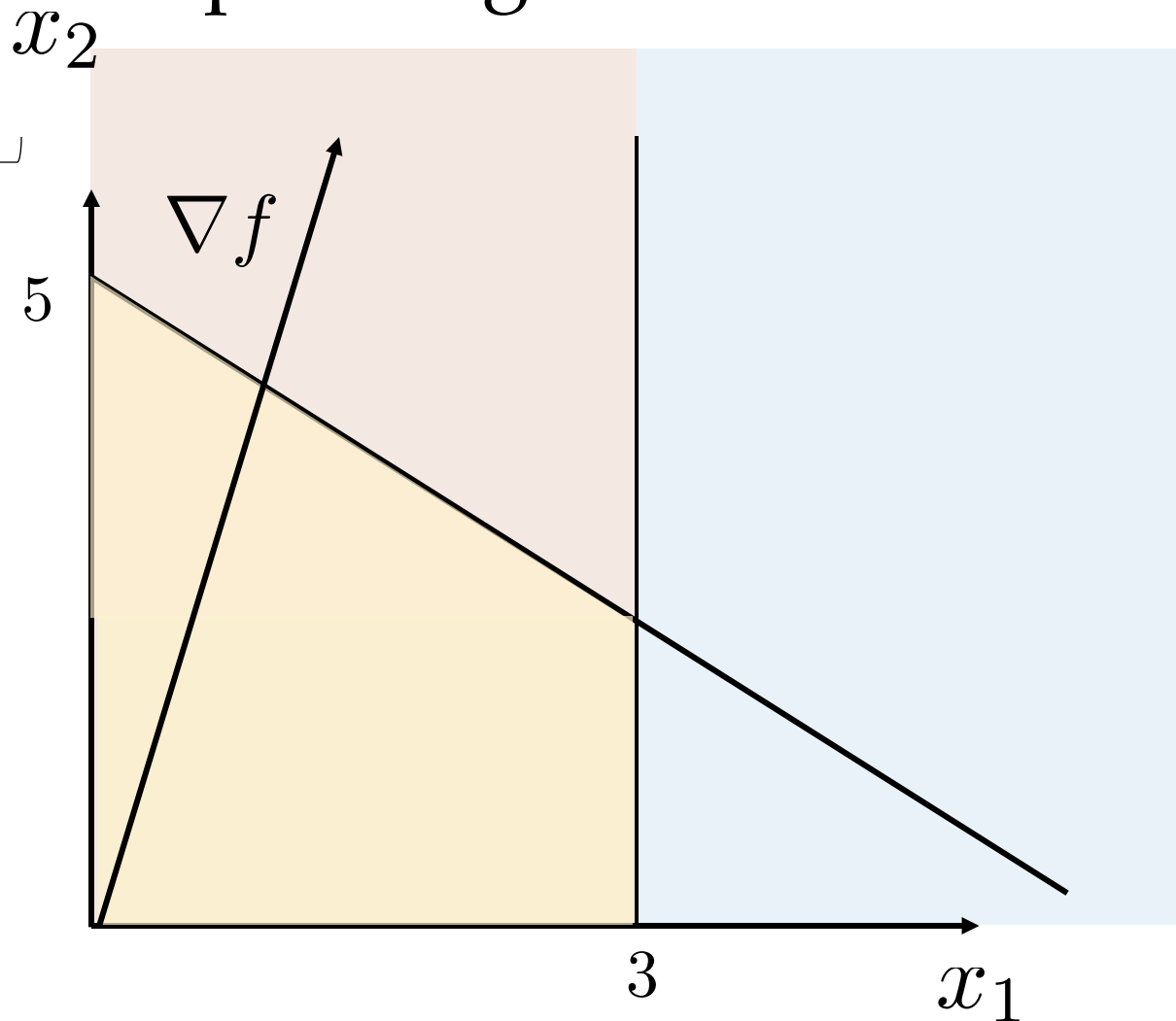
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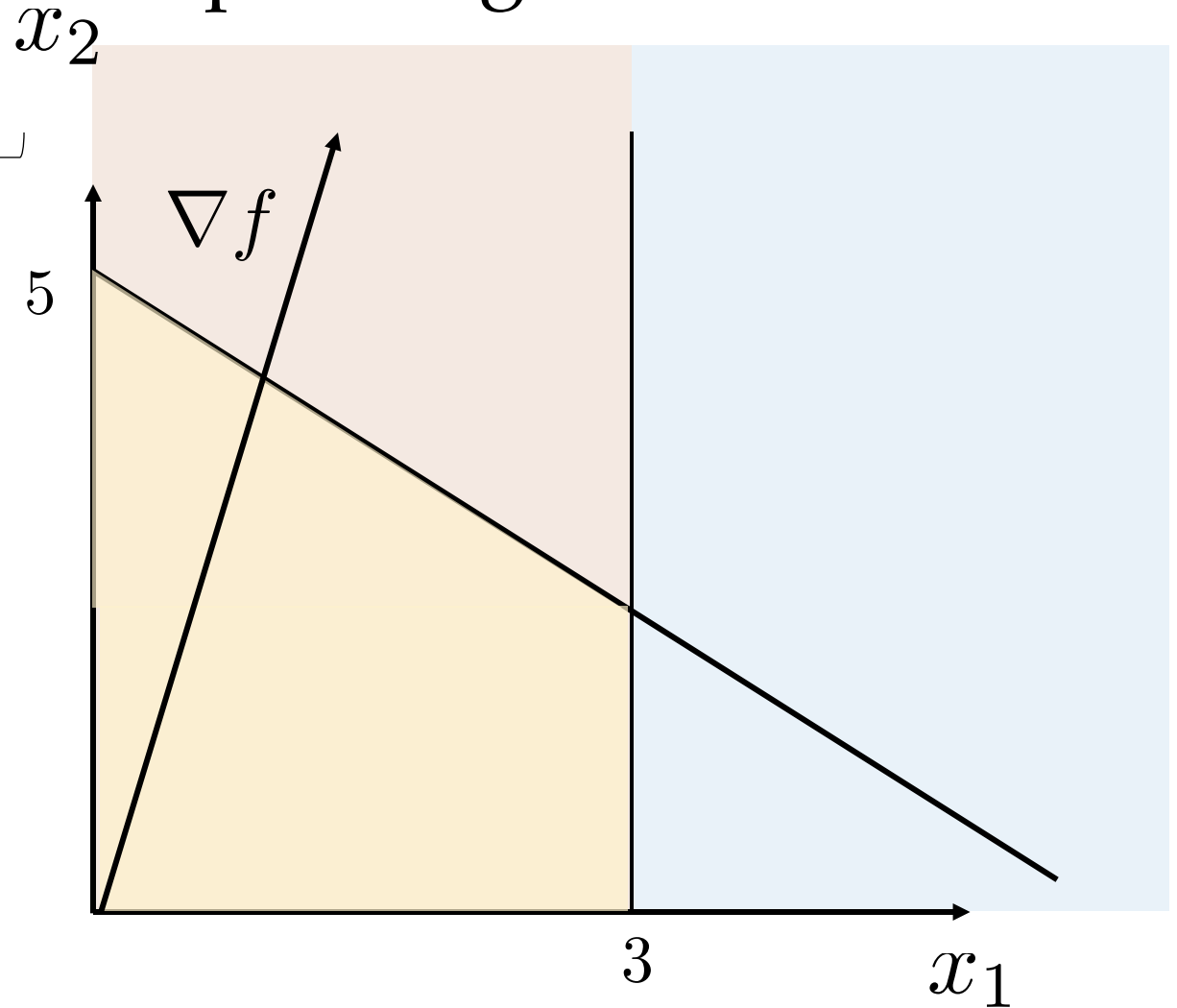
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Intuition Behind Simplex Algorithm

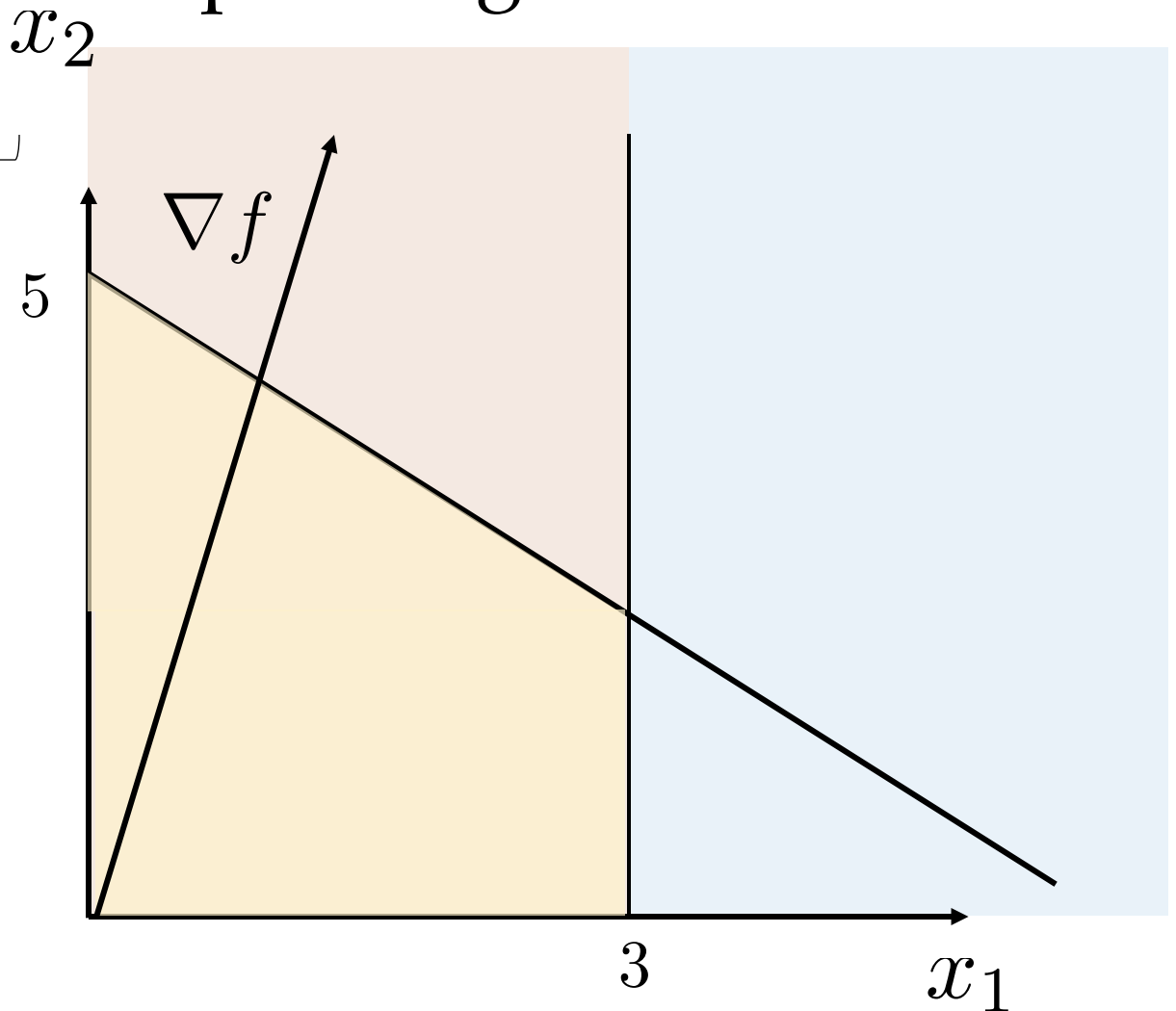
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Intuition: If we are not in a corner, there is always a direction we can go to improve the function.



Intuition Behind Simplex Algorithm

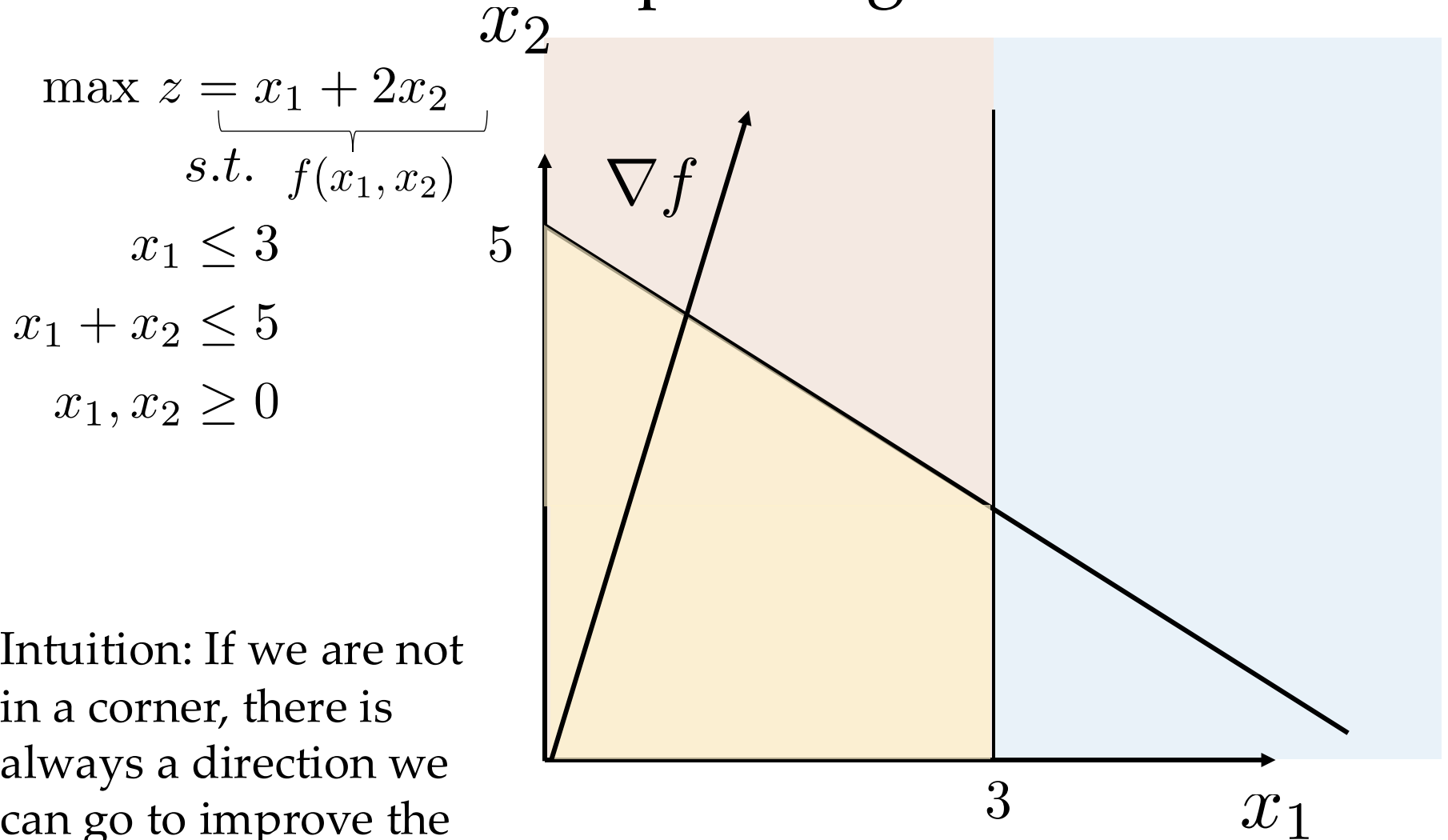
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If there is no adjacent corner that improves f , it is the optimum

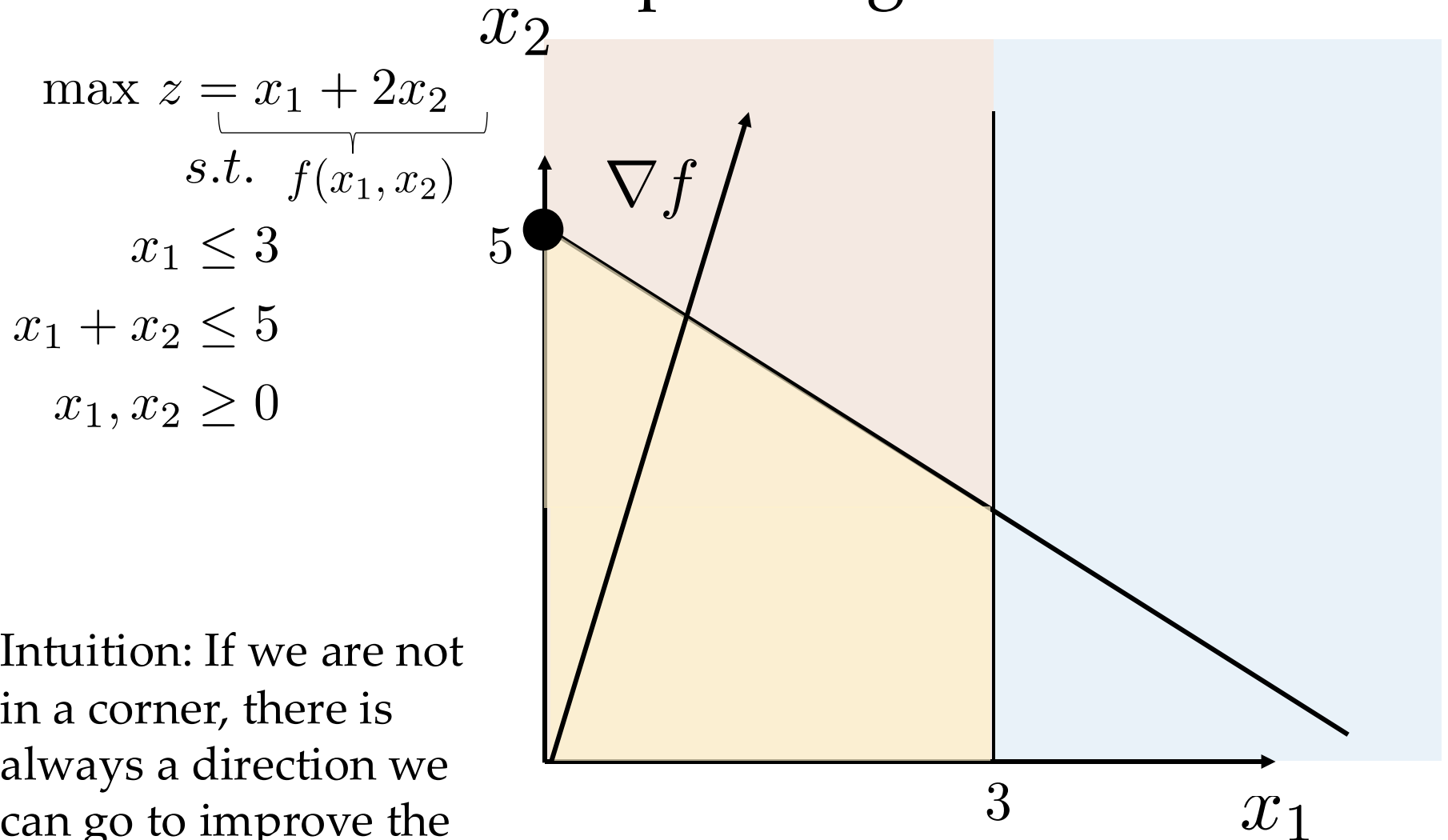
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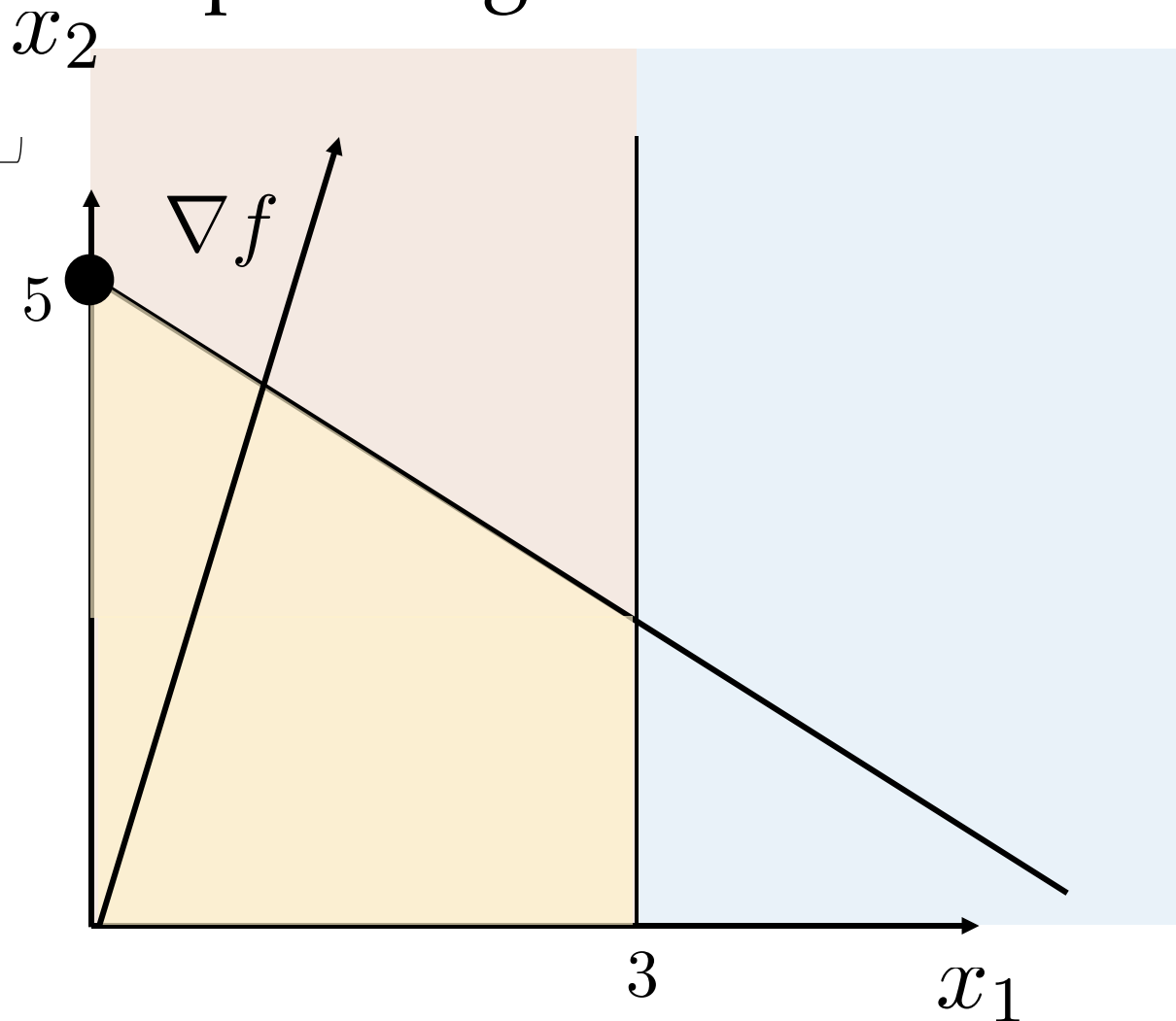
$$s.t. \quad f(x_1, x_2)$$

$$x_1 \leq 3$$

$$x_1 + x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

When does this not hold?



Constrained Optimization: General Case

- We have arbitrary non-linear objective and constrain functions, f , g .
- We use a method called *Lagrange Multipliers*

Example: two variables

- We can represent a constraint as a curve $g(x_1, x_2) = 0$



Example: two variables

- To stay on the curve we need to follow the tangent.



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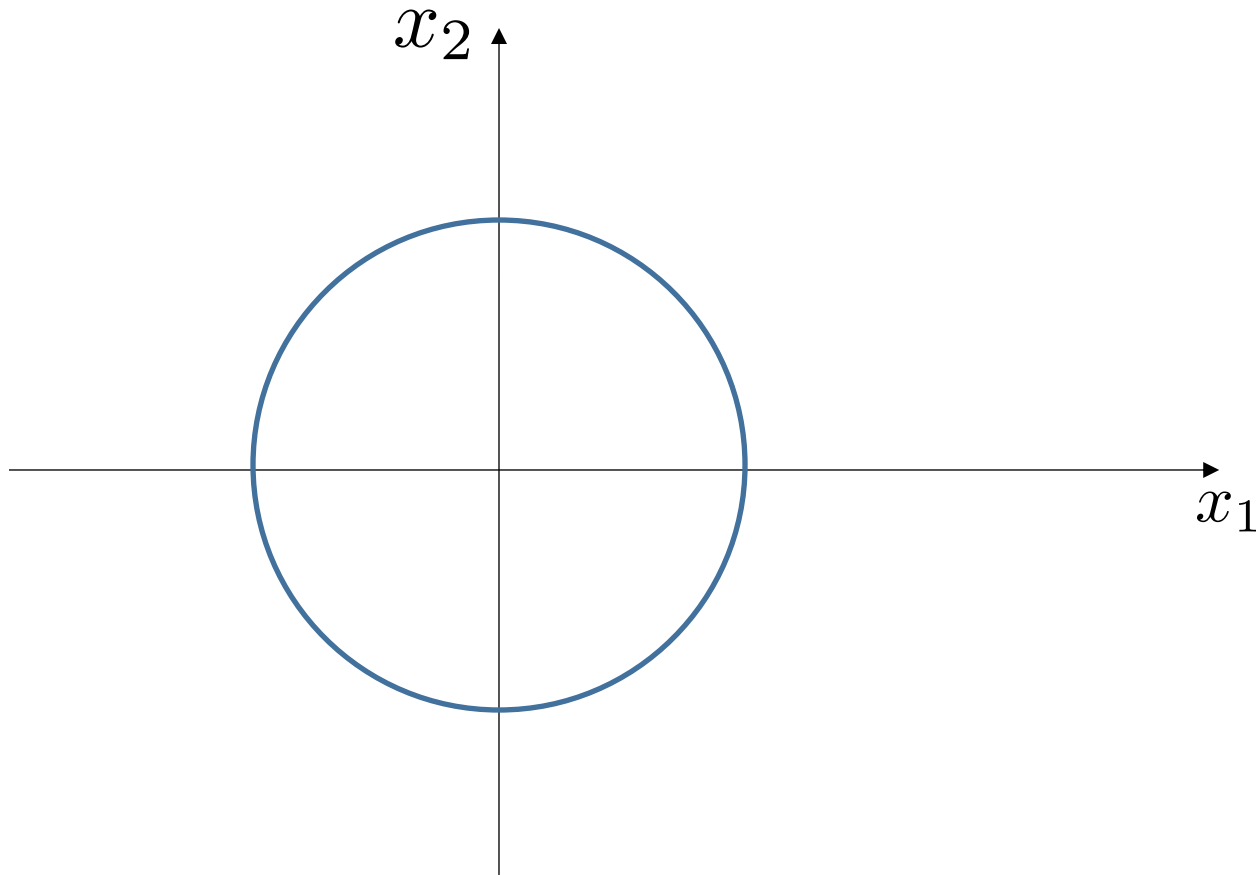


Example: Circle

$$g(x_1, x_2) = x_1^2 + x_2^2 = 1$$

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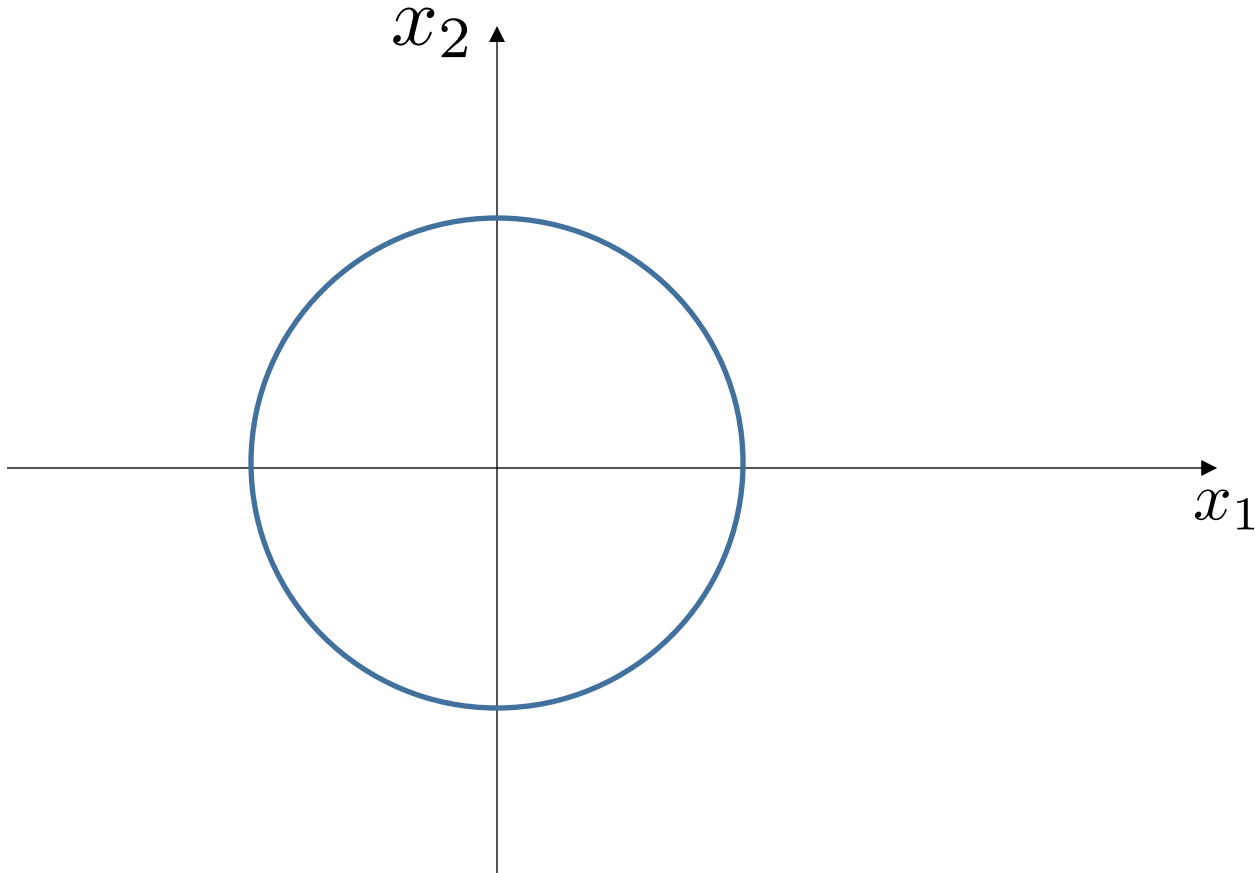
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Example: Circle

$$g(x_1, x_2) = x_1^2 + x_2^2 = 1$$

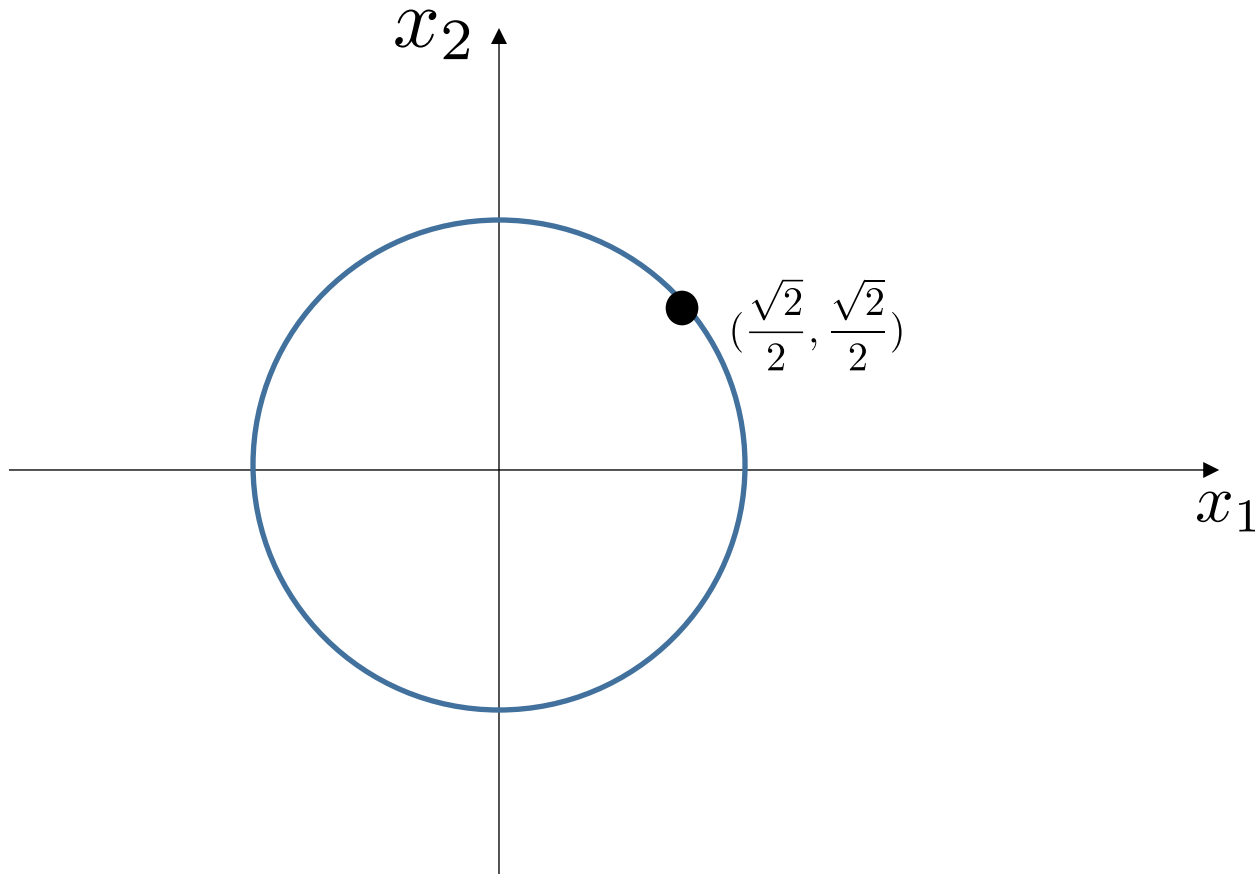
$$\nabla g(x_1, x_2) = (2x_1, 2x_2)$$



Example: Circle

$$g(x_1, x_2) = x_1^2 + x_2^2 = 1$$

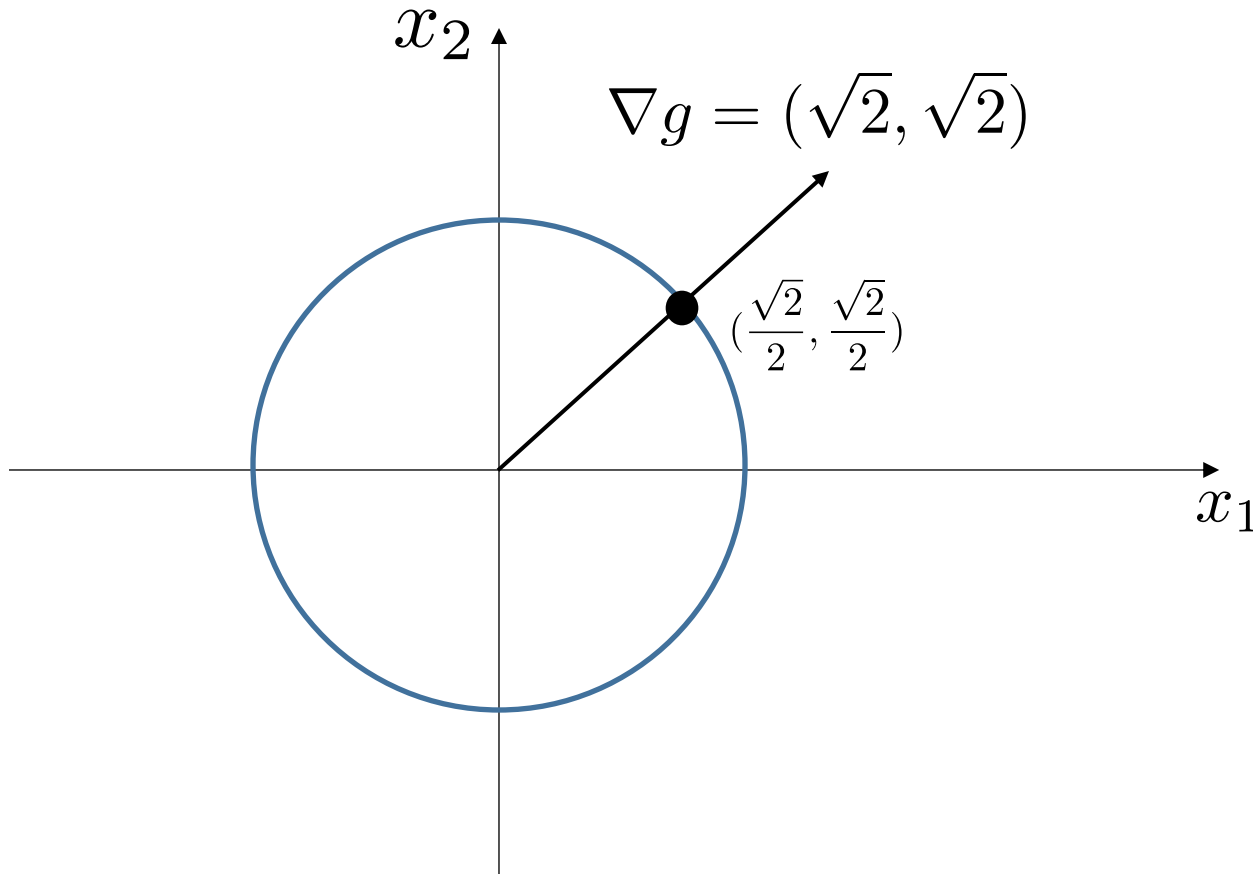
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Example: Circle

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$$\nabla g(x_1, x_2) = (2x_1, 2x_2)$$



Example: two variables

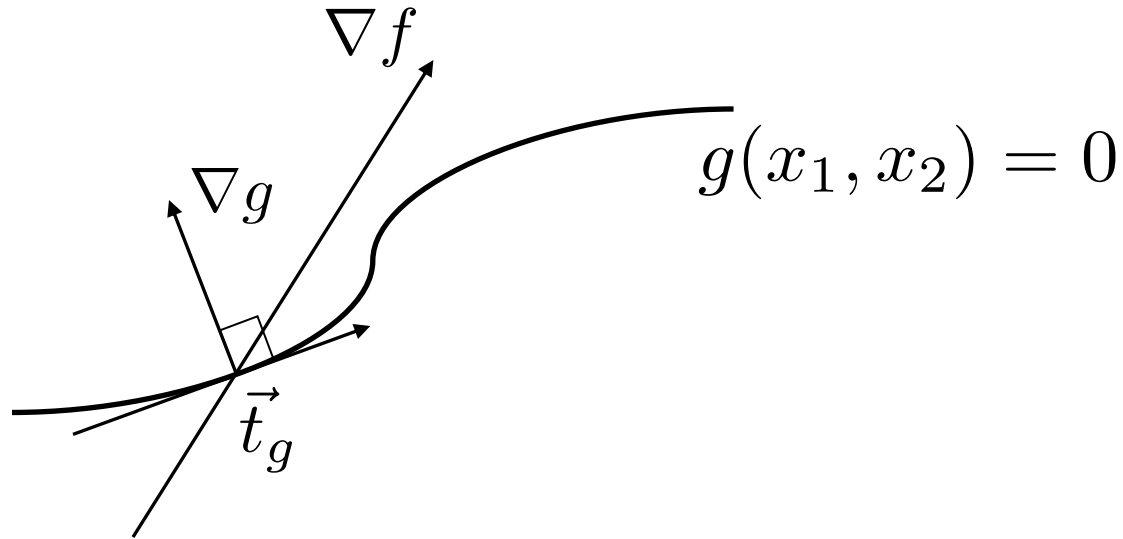
- To stay on the curve we need to follow the tangent.



$$\nabla g \cdot \vec{t}_g = 0$$

Example: two variables

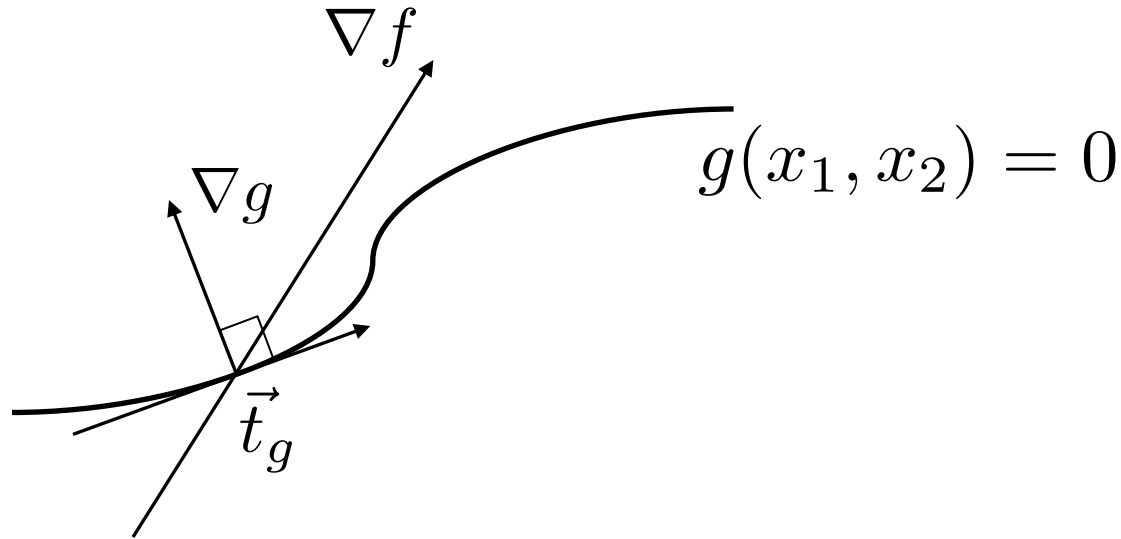
- To stay on the curve we need to follow the tangent.



$$\nabla g \cdot \vec{t}_g = 0$$

Example: two variables

- Generally, motion along the constraint curve will increase or decrease $f(x, y)$

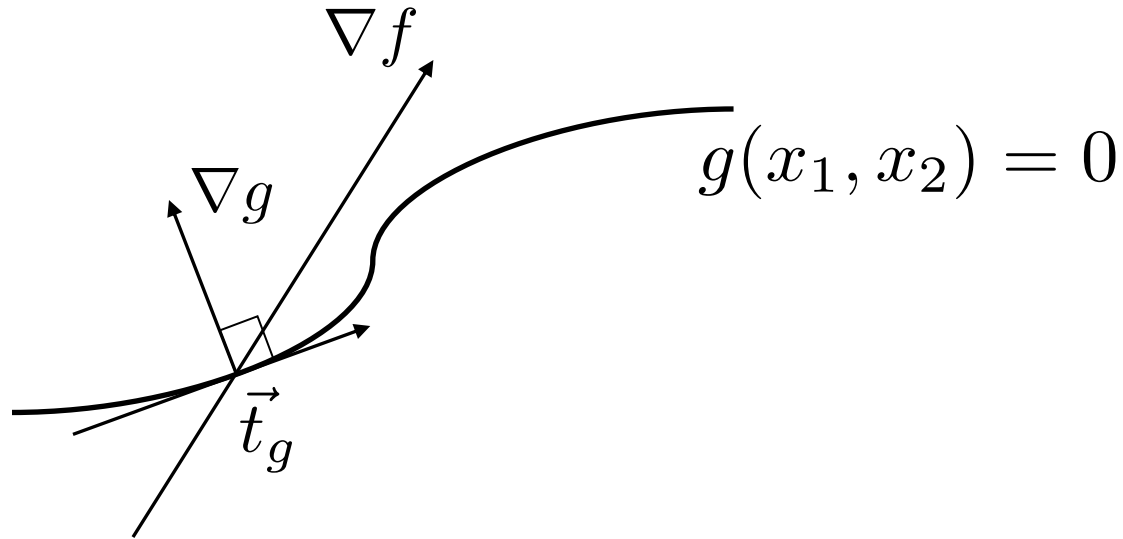


$$\nabla g \cdot \vec{t}_g = 0$$

$$\nabla f \cdot \vec{t}_g \neq 0$$

Example: two variables

- At the optimum:

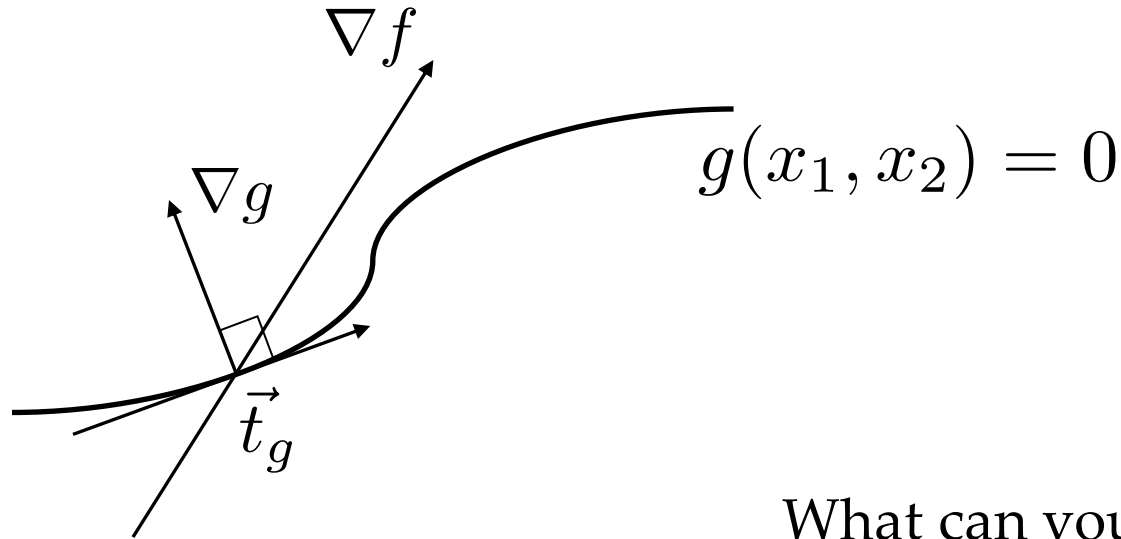


$$\nabla g \cdot \vec{t}_g = 0$$

$$\nabla f \cdot \vec{t}_g = 0$$

Example: two variables

- At the optimum:



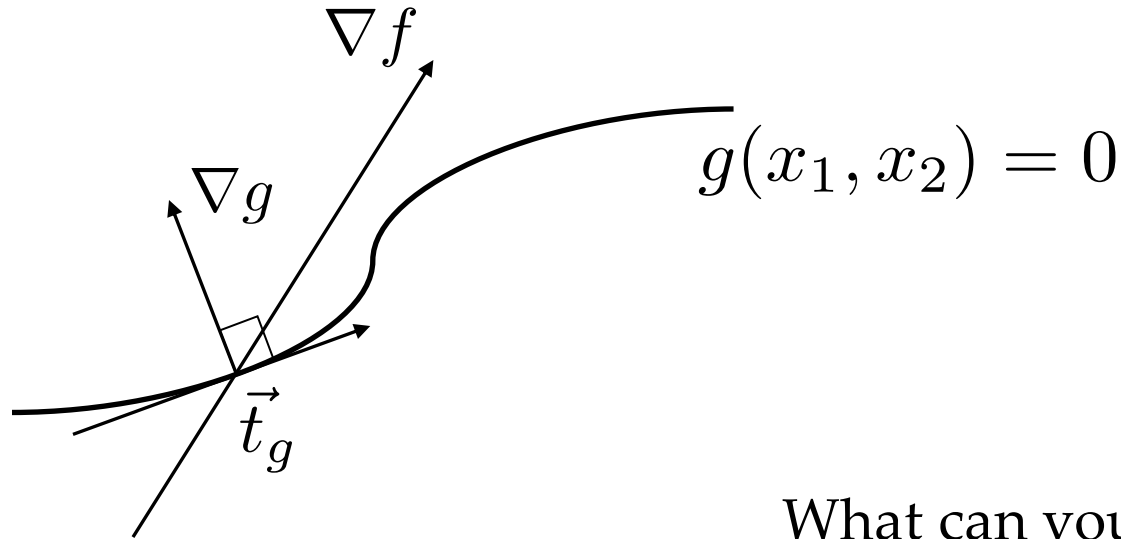
What can you conclude about $\nabla f, \nabla g$?

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Example: two variables

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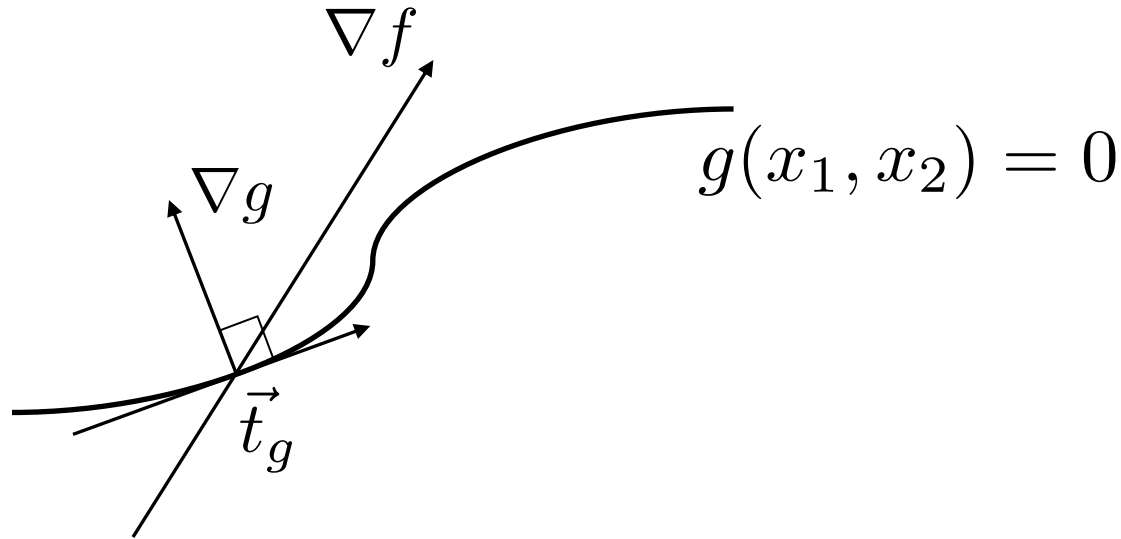
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$$\nabla f \cdot \vec{t}_g = 0$$

$$\nabla f + \lambda \nabla g = 0$$

Example: two variables

- At the optimum:



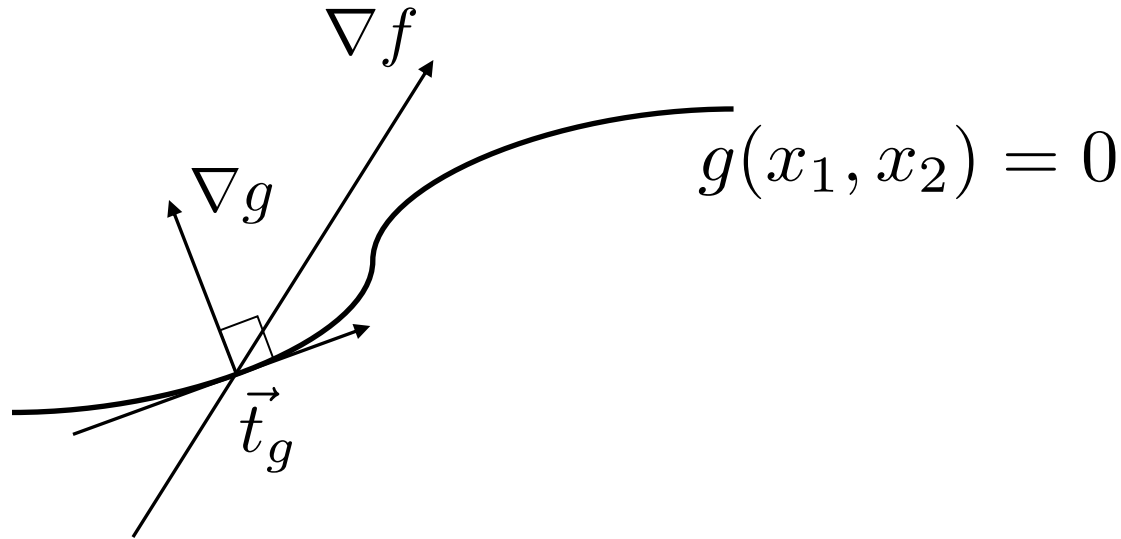
Define the set C_f of points (x_1, x_2)
where:

$$g(x_1, x_2) = 0$$

$$\nabla f + \lambda \nabla g = 0$$

Example: two variables

- At the optimum:



Define the set C_f of points (x_1, x_2)
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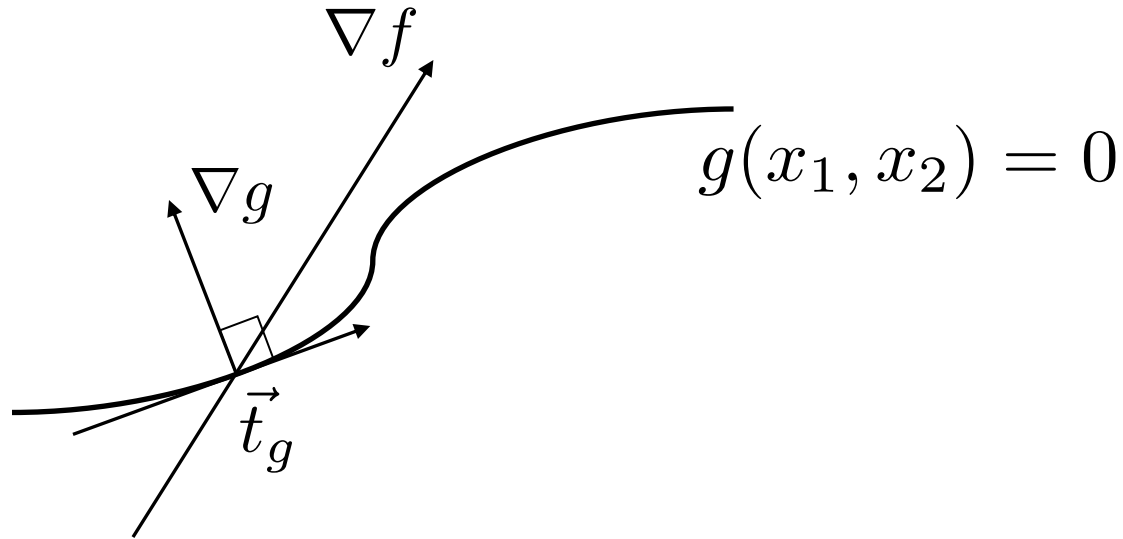
$$g(x_1, x_2) = 0$$

These are the extrema points!

$$\nabla f + \lambda \nabla g = 0$$

Example: two variables

- At the optimum:



Let

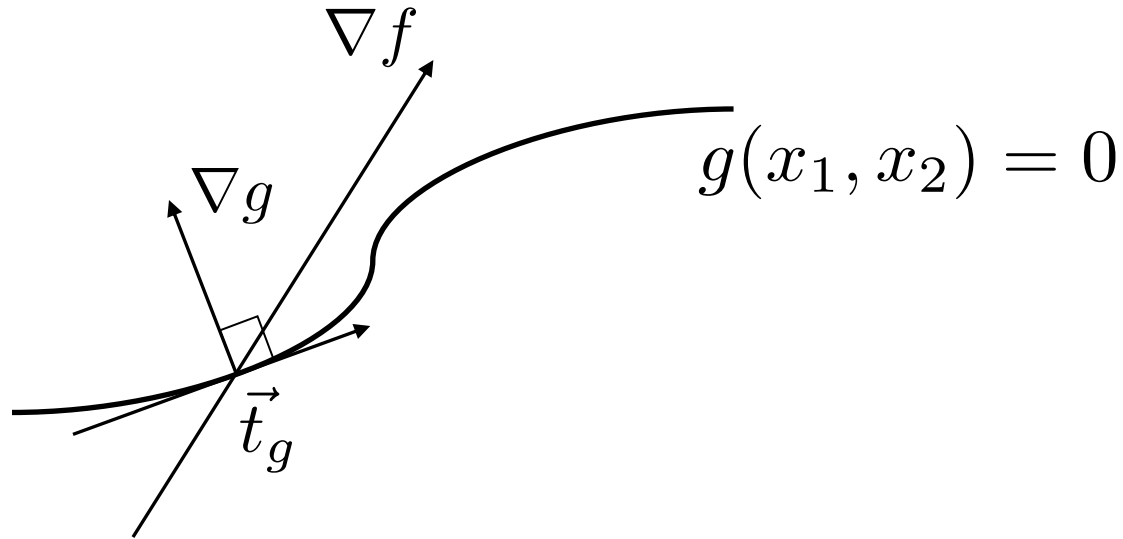
$$F(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

Find x_1, x_2, λ so that:

$$\nabla F = 0$$

Example: two variables

- At the optimum:



Let

$$F(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

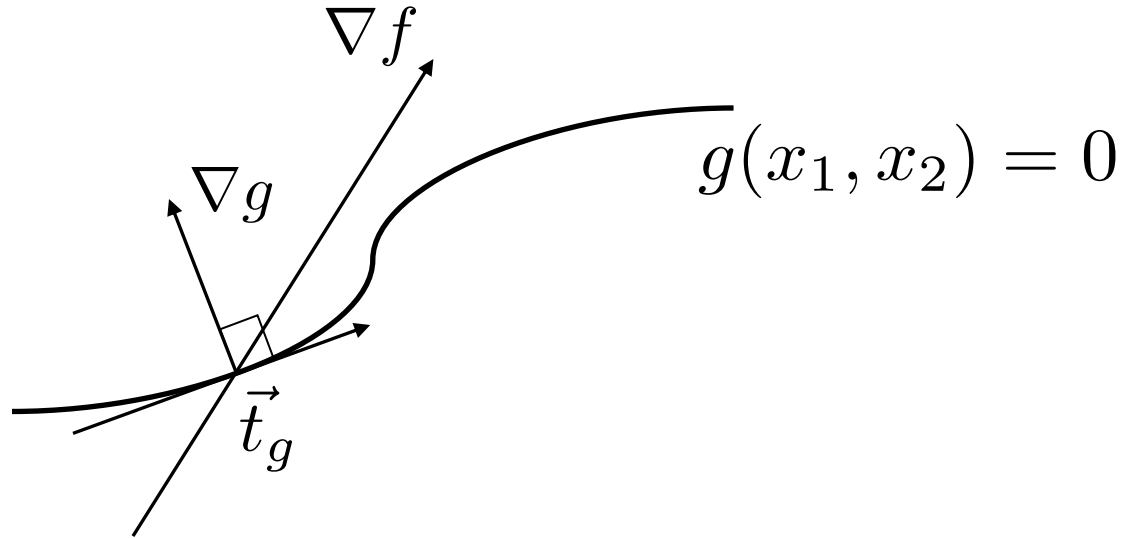
$$\frac{\partial F}{\partial x_1} = 0$$

$$\frac{\partial F}{\partial x_2} = 0$$

$$\frac{\partial F}{\partial \lambda} = 0$$

Example: two variables

- At the optimum:



Let

$$F(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

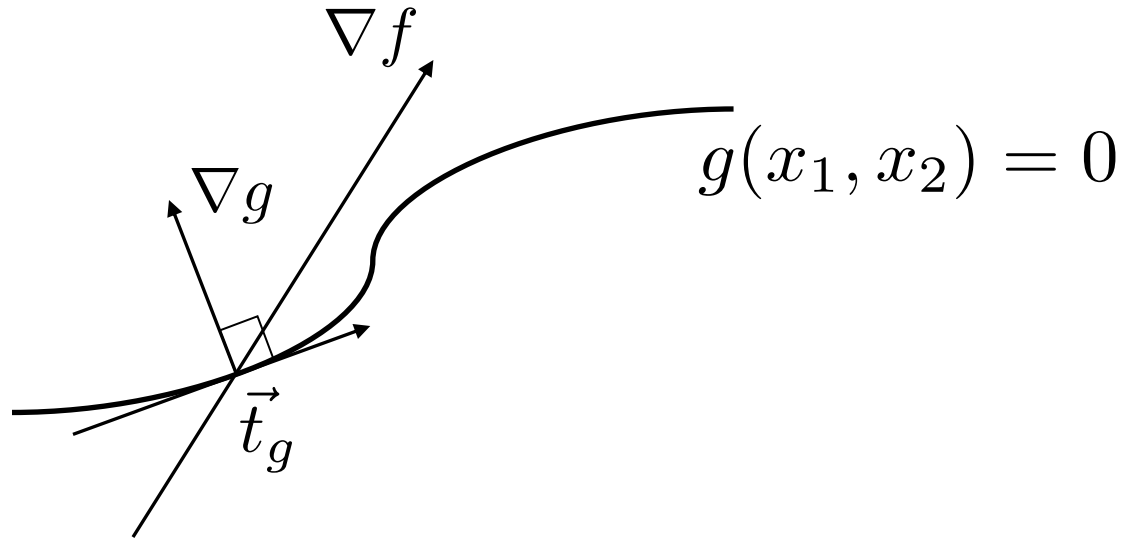
$$\frac{\partial F}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} = 0$$

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$$\frac{\partial F}{\partial \lambda} = g(x_1, x_2) = 0$$

Example: two variables

- At the optimum:



Define the set C_f of points (x_1, x_2)
where:

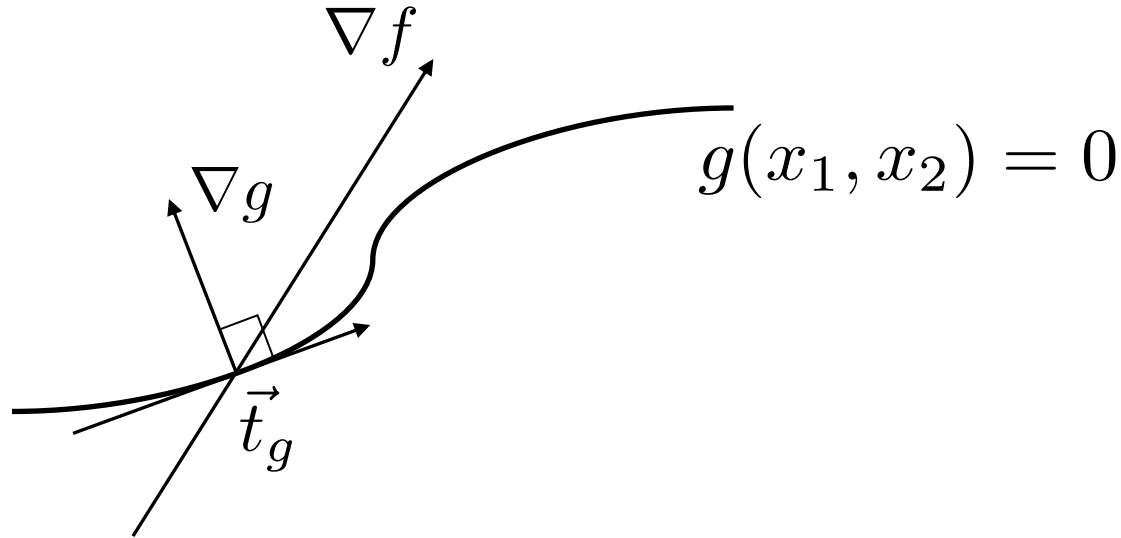
$$g(x_1, x_2) = 0$$

These are the extrema points!

$$\nabla f + \lambda \nabla g = 0$$

Example: two variables

- At the optimum:



Let

$$F(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

F : Lagrangian

$$\frac{\partial F}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} = 0$$

$$\frac{\partial F}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} = 0$$

$$\frac{\partial F}{\partial \lambda} = g(x_1, x_2) = 0$$

λ : Lagrange multiplier

Example

- Find the extrema values of the function $f(x_1, x_2) = x_1x_2$, subject to the constraint:

$$g(x_1, x_2) = \frac{x_1^2}{8} + \frac{x_2^2}{2} - 1 = 0$$

$$F(x_1, x_2)?$$

Example

- Find the extrema values of the function $f(x_1, x_2) = x_1x_2$, subject to the constraint:

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$$F(x_1, x_2) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

$$\nabla F(x_1, x_2, \lambda) = \begin{bmatrix} x_2 + \frac{\lambda x_1}{4} \\ x_1 + \lambda x_2 \\ \frac{x_1^2}{8} + \frac{x_2^2}{2} - 1 \end{bmatrix} = 0$$

Example

$$\nabla F(x_1, x_2, \lambda) = \begin{bmatrix} x_2 + \frac{\lambda x_1}{4} \\ x_1 + \lambda x_2 \\ \frac{x_1^2}{8} + \frac{x_2^2}{2} - 1 \end{bmatrix} = 0$$

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$$\left. \begin{array}{l} x_2 = -\frac{\lambda x_1}{4} \\ x_1 = -\lambda x_2 \\ x_1^2 = 8 - 4x_2^2 \end{array} \right\}$$

Example

$$\nabla F(x_1, x_2, \lambda) = \begin{bmatrix} x_2 + \frac{\lambda x_1}{4} \\ x_1 + \lambda x_2 \\ \frac{x_1^2}{8} + \frac{x_2^2}{2} - 1 \end{bmatrix} = 0$$

$$\left. \begin{array}{l} x_2 = -\frac{\lambda x_1}{4} \\ x_1 = -\lambda x_2 \\ x_1^2 = 8 - 4x_2^2 \end{array} \right\} \Rightarrow \left. \begin{array}{l} x_2 = \frac{\lambda^2 x_2}{4} \\ x_1 = -\lambda x_2 \\ x_1^2 = 8 - 4x_2^2 \end{array} \right\}$$

Example

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Example

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Solution

$$\nabla F(x_1, x_2, \lambda) = \begin{bmatrix} x_2 + \frac{\lambda x_1}{4} \\ x_1 + \lambda x_2 \\ \frac{x_1^2}{8} + \frac{x_2^2}{2} - 1 \end{bmatrix} = 0$$

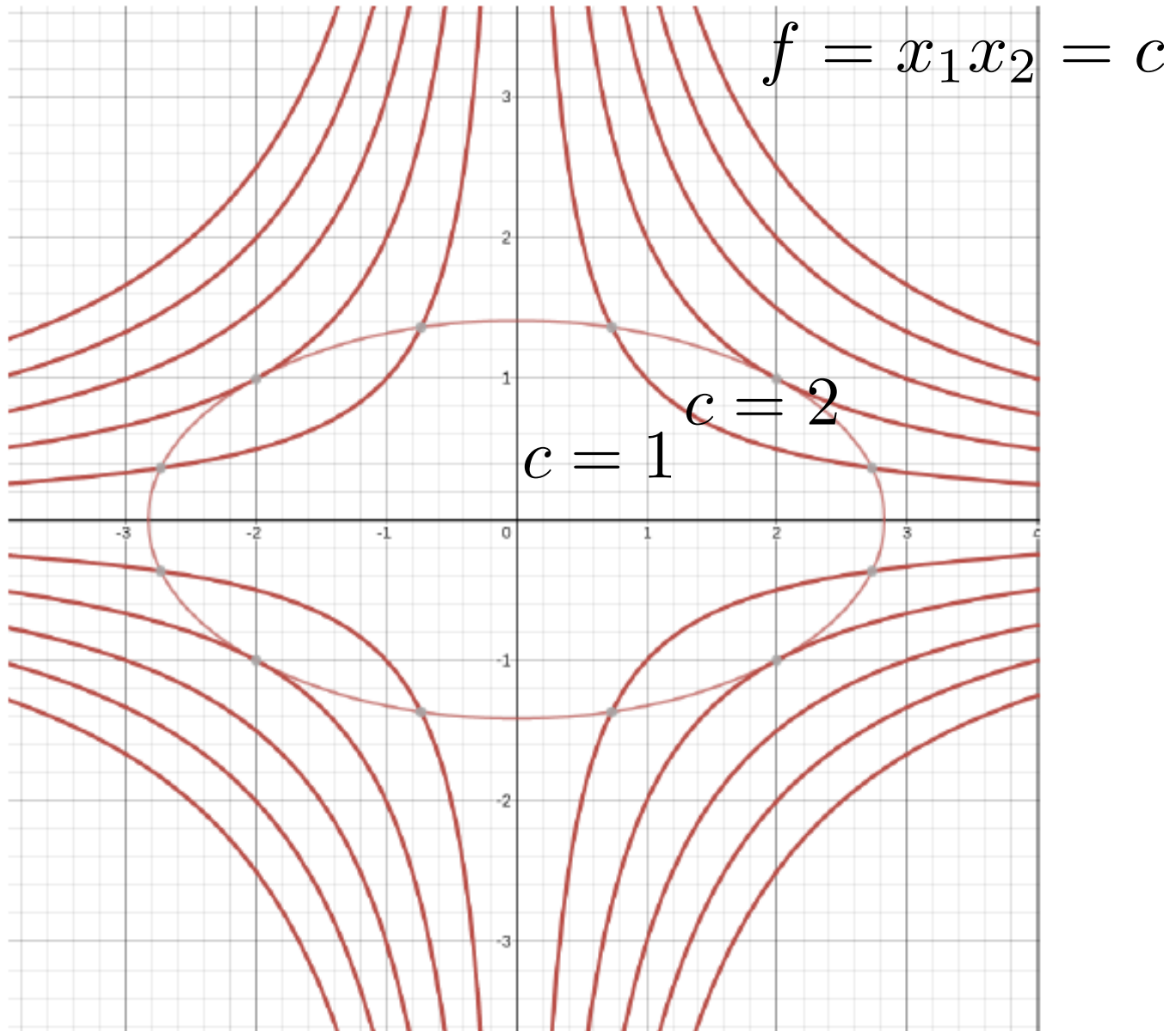
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$$\lambda = \pm 2$$

$$x_1 = \pm 2$$

$$x_2 = \pm 1$$

Solution



Application: Inverse Kinematics

- Given a desired position of the robot's end-effector, find the values of the joint angles that are within their limits and avoid collisions.

