Lecture 3: Probability Theory Refresher

[Resources: 6.825 Techniques in Artificial Intelligence, MIT OCW
16-811 Math Fundamentals for Robotics, CMU]

CSCI 545 Introduction to Robotics
Instructor: Stefanos Nikolaidis

Motivation

- Why do we need probability?
  - To understand how likely an outcome is!

Motivation

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Is it deterministic or non-deterministic?

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Is it deterministic or non-deterministic?

From a Bayesian perspective, it doesn't matter!

Probability Theory

- It is a logic; a language that speaks about likelihood of ev ents.

- We can start with a universe of atomic events: things that could happen or ways that the world could be

- Then, a probability distribution is a function that maps events into a range between zero and one: P: events→[0, 1]

Probability Theory Axioms

- P(True) = 1 = P(U)

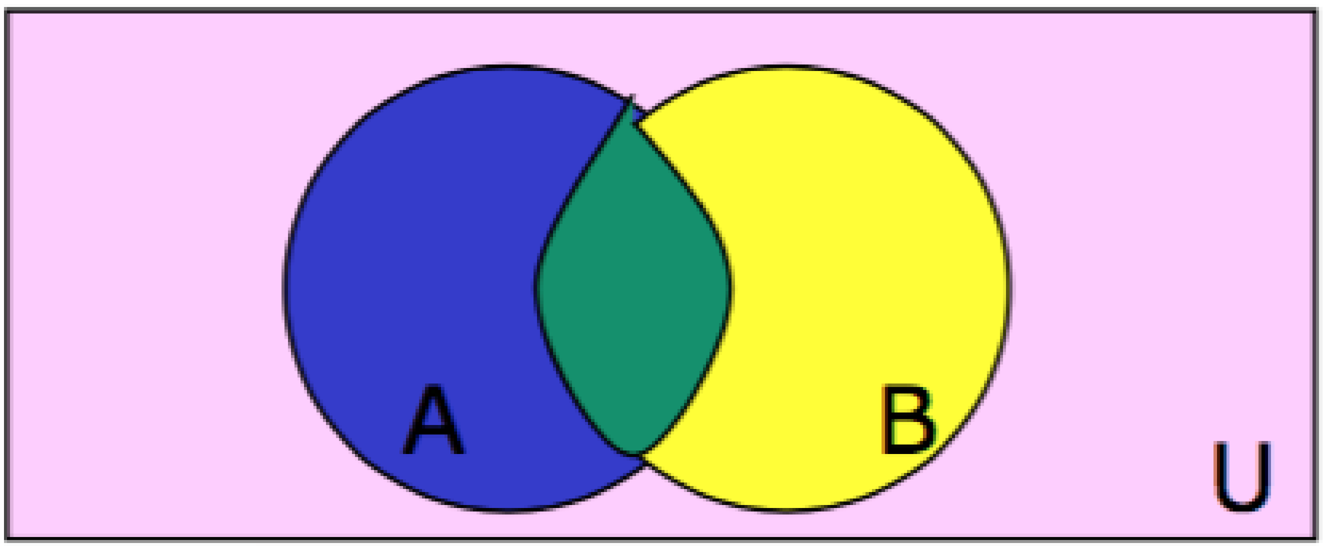
The probability of True is one. It is the probability of the universe, the probability that something in this realm of discussion that we have available to use is one.

- P(False) = 0 = P(null)

The probability of False is 0. In terms of atomic events, False is the empty set.

Probability Theory Axioms

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



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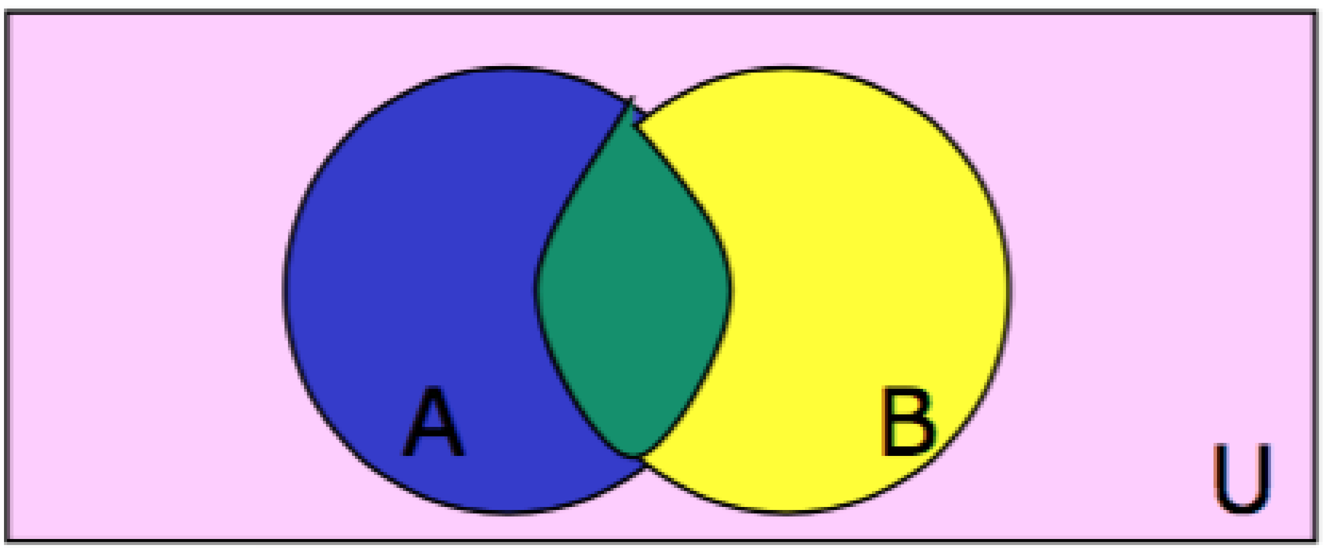


Random Variables

- A random variable: it can be considered as a mapping from a discrete domain to a range from 0 to 1.

- It sums to 1 over the domain

- Example RV: Raining
  - Domain: True / False
  - Raining (True) = 0.2

Random Variables

- Raining (True) = 0.2
- P(Raining = True) = 0.2
- P(Raining = False) = ?

Random Variables

- Raining (True) = 0.2
- P(Raining = True) = 0.2
- P(Raining = False) = **0.8**

Multiple Random Variables

	Light Off	Light On
Battery Low	0.04	0.06
Battery High	0.01	0.89

Multiple Random Variables

	Light Off	**Light On**
Battery Low	0.04	0.06
Battery High	0.01	0.89

P(Battery = low) = 0.1
P(Battery = high) = 0.9

Multiple Random Variables

	Light Off	**Light On**
Battery Low	0.04	0.06
Battery High	0.01	0.89

P(Light = off) = ?

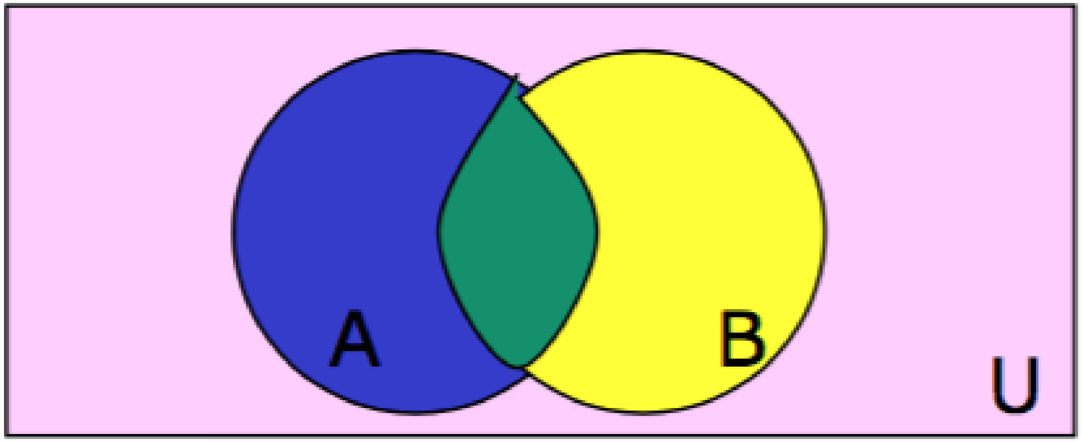
Multiple Random Variables

	Light Off	**Light On**
Battery Low	0.04	0.06
Battery High	0.01	0.89

P(Light = off) = **0.05**

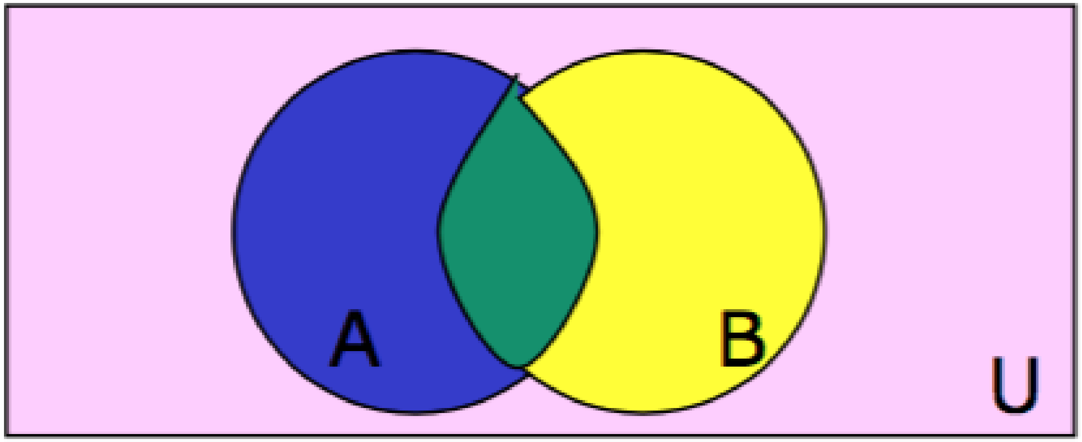
Conditional Probability

- P (A | B)?



Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



We only consider the part of the world, where B is True.

Conditional Probability

	Light Off	Light On
Battery Low	0.04	0.06
Battery High	0.01	0.89

$$P(\text{Battery-Low}|\text{Light-Off}) =$$

Conditional Probability

	Light Off	**Light On**
Battery Low	0.04	0.06
Battery High	0.01	0.89

$$P(\text{Battery-Low}|\text{Light-Off}) = \frac{0.04}{0.05} = 0.8$$

Posterior probability

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(\text{State}|\text{Observation}) = \frac{P(\text{Observation}|\text{State})P(\text{State})}{P(\text{Observation})}$$

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Independent Events

- Knowing that B is true doesn’t give us any more information about A.

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$

Independent Events

- Are temperature and time of the day independent?
- Are the state of the battery and the whether the motor is damaged or not independent?

Conditional Independence

- A and B are conditionally independent given C, iff the probability of A given B and C is equal to the probability of A given C:

$$P(A|B,C) = P(A|C)$$

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Conditional Independence: Example

- Random Variables:
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  - Light Off (L)
  - No Motion (N)
- What is the relationship of low battery, light off and no motion?

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- Does seeing the robot with a light off tell us whether it is going to move?

Conditional Independence: Example

- Random Variables:
  - Low Battery (B)
  - Light Off (L)
  - No Motion (N)
- What is the relationship of low battery, light off and no motion?
- Does seeing the robot with a light off tell us whether it is going to move?
- What if we know that the battery is low? Does seeing a light off give us some additional information?

Incorporating Multiple Sources of Information

- Random Variables:
  - Low Battery (B)
  - Light Off (L)
  - No Motion (N)

$$P(B|L,N) = \frac{P(L,N|B)P(B)}{P(L,N)}$$

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Incorporating Multiple Sources of Information

- Random Variables:
  - Low Battery (B)
  - Light Off (L)
  - No Motion (N)

$$P(L, N) = P(L|B)P(N|B)P(B) + P(L|-B)P(N|-B)P(-B)$$

Continuous Random Variables

- Example: Position of the robot in the room

- A continuous random variable possesses a probability density function (PDF), which always integrates to 1:

$$\int p(x)dx = 1$$

Normal Distribution

- Normal distributions play a major role in this lecture. We will abbreviate them as

$$N(x; \mu, \sigma^2)$$

$$p(x) = (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Linear Transformations

- if x and y are independent and $x \sim \mathcal{N}(\mu_x, Q_x), y \sim \mathcal{N}(\mu_y, Q_y)$ with $z = Ax + By + \mathrm{c}$, with c a constant, then:

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- z is also a normal random variable with

  $\text{z} \sim \mathcal{N}\left(A\mu_x + B\mu_y + c, A^T Q_x A + B^T Q_y B\right)$

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  $\text{z} \sim \mathcal{N}(A\mu_x + B\mu_y + c, A^T Q_x A + B^T Q_y B)$

- Example: Let $x \sim \mathcal{N}(0,1)$ and $\text{y} \sim \mathcal{N}(0,4)$ and $z = x + 2y + 3$ then:

  $\text{z} \sim \mathcal{N}(3, 1 * 1 * 1 + 2 * 4 * 2) = \mathcal{N}(3, 17)$

Probability Axioms for Continuous Variables

$$p(x) = \sum_y p(x|y)p(y) \qquad \text{(discrete)}$$

$$p(x) = \int_y p(x|y)p(y)dy \qquad \text{(continuous)}$$

Probability Axioms for Continuous Variables

$$p(x|y) = \frac{p(y|x)p(x)}{\sum_{x'} p(y|x')p(x')} \qquad \text{(discrete)}$$

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Probability Axioms for Continuous Variables

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$$p(x|y) = \eta p(y|x)p(x)$$

Expectation of Random Variable

$$E[X] = \sum_x x p(x)$$ (discrete)

$$E[X] = \int_x x p(x) dx$$ (continuous)

$$E[aX + b] = aE[X] + b$$

Example

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- Expected Value = (-$3) * (1/3) + ($2) * (1/3) + ($5) * (1/3) Expected Value = (-$1) + ($2/3) + ($5/3)

  Expected Value = $4/3

Example

- Let's assume that the time the battery of a robot lasts follows a probability distribution with density function: f(x) = 0.02 x for $0 \leq x \leq 50$, 0 otherwise. What's the expected battery life of the robot?

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$$E(X) = \frac{0.02}{3} \cdot (50^3 - 0^3)$$

$$E(X) = \frac{0.02}{3} \cdot 125000$$

$$E(X) = 833.33$$

Example

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$$E(X) = \int_0^1 x dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

Example

- Q: We Let $Z \sim N(0,1)$. Find E(Z).
- A: Z has range (-inf, inf) and density $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$

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$$E(Z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z e^{-\frac{z^2}{2}} dz = -\frac{1}{\sqrt{2\pi}} z e^{-\frac{z^2}{2}} \Big|_{-\infty}^{\infty} = 0$$

Random Vectors

- Normalization: $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\mathbf{x}) dx_1 dx_2 \ldots dx_n = 1$$

- Mean: $$\bar{\mathbf{x}} = E[\mathbf{x}] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbf{x} f(\mathbf{x}) dx_1 dx_2 \ldots dx_n$$

- Covariance:

$$\Sigma = [\Sigma_{ij}] = E[(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})^T] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})^T f(\mathbf{x}) dx_1 dx_2 \ldots dx_n$$

Example

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$$C(\mathbf{x}) = E\left\{ \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ x_3 - \mu_3 \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 & x_3 - \mu_3 \end{bmatrix} \right\}$$

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$$C(\mathbf{x}) = E\left\{ \begin{bmatrix} (x_1 - \mu_1)^2 & (x_1 - \mu_1)(x_2 - \mu_2) & (x_1 - \mu_1)(x_3 - \mu_3) \\ (x_2 - \mu_2)(x_1 - \mu_1) & (x_2 - \mu_2)^2 & (x_2 - \mu_2)(x_3 - \mu_3) \\ (x_3 - \mu_3)(x_1 - \mu_1) & (x_3 - \mu_3)(x_2 - \mu_2) & (x_3 - \mu_3)^2 \end{bmatrix} \right\}$$

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$$C(\mathbf{x}) = \begin{bmatrix} V(x_1) & Cov(x_1, x_2) & Cov(x_1, x_3) \\ Cov(x_1, x_2) & V(x_2) & Cov(x_2, x_3) \\ Cov(x_1, x_3) & Cov(x_2, x_3) & V(x_3) \end{bmatrix}$$

Stationarity

- Stationarity: Parameters, e.g., mean, covariance, are time-shift invariant: $E[x(t)] = \mu \ \forall t$

- Are the following processes z stationary:

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  $z = \mathrm{t} + w, \ w \sim \mathcal{N}(0,1)$

Markov Chains

- A set of trials with possible outcomes, $e_1, e_2, \ldots$ is called a Markov chain if the probabilities of sampled sequences are defined by:
- $P\big(\{e_{j1}, \ldots e_{jT}\}\big) = P(e_{j1})\, P(e_{j2} |\, e_{j1}) \ldots P(e_{jT} |\, e_{jT-1})$

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- The probabilities of transitioning from outcome e_{k-1} to e_k are defined with a transition matrix

$$T = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \vdots & & \\ p_{31} & p_{32} & \cdots \end{bmatrix}$$

Example

- Assume three possible system states, e_1, e_2 and e_3:
  - e_1 = Battery full
  - e_2 = Battery low
  - e_3 = Battery empty

$$T = \begin{bmatrix} a & 1-a & 0 \\ 0 & b & 1-b \\ 0 & 0 & 1 \end{bmatrix}$$

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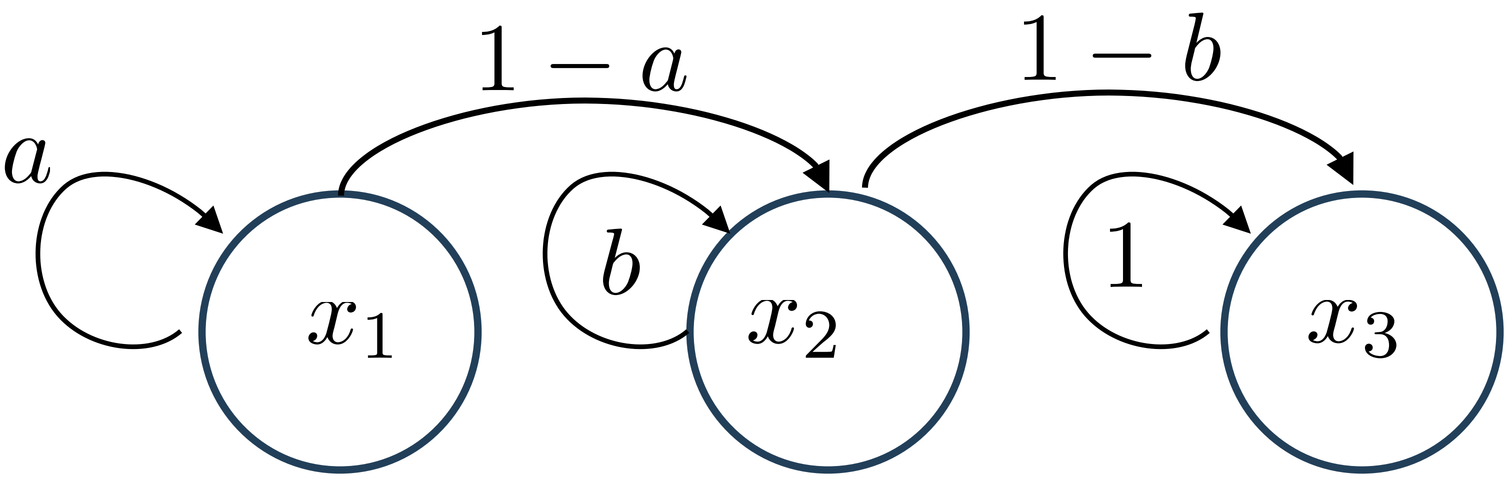
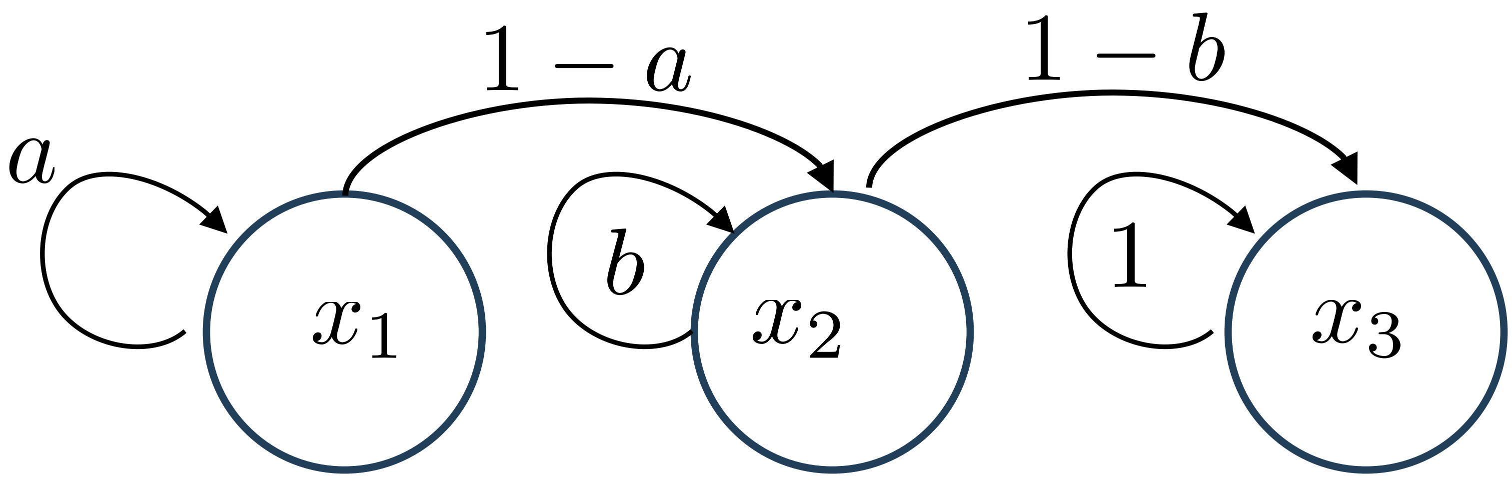


Example

- What is the probability of the sequence (“battery full”, ”battery low”, “battery low”) given that the robot starts with battery full?



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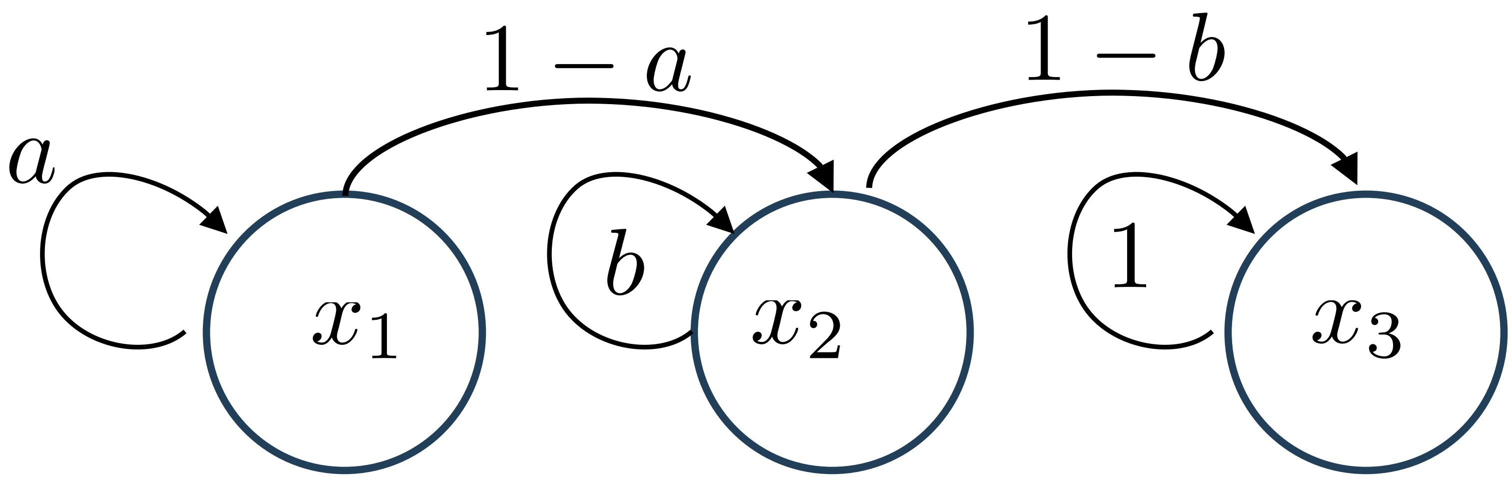
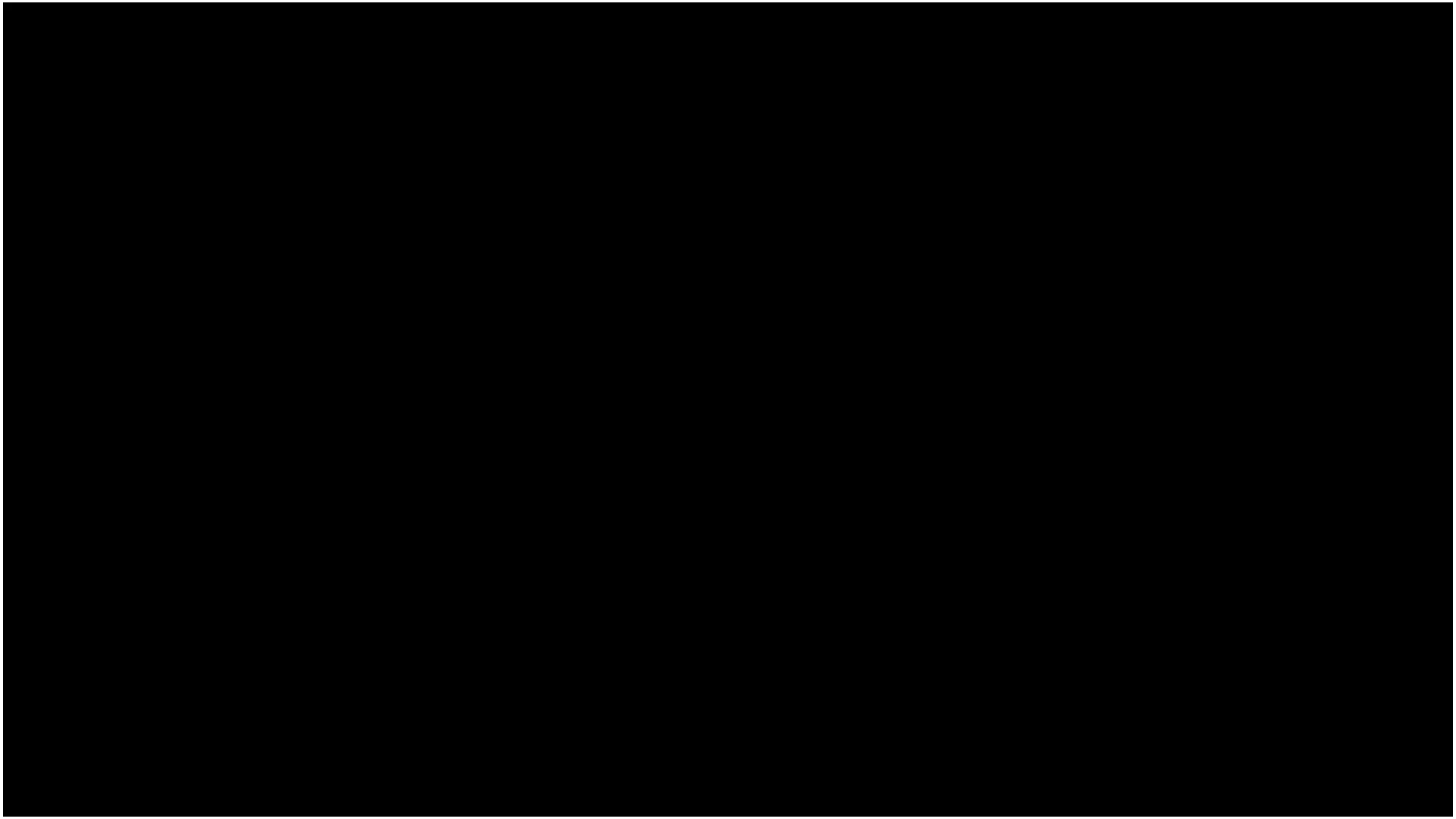
$$P(\{e_1, e_2, e_2\}) = P(e_1)P(e_2|e_1)P(e_2|e_2)$$
$$= a * (1 - a) * b$$



Reasoning over uncertainty



State

- A *state* captures all the information that the robot needs o make decisions. It may change over time such as its location.
- The are two types of interaction that the robot can have with the environment.

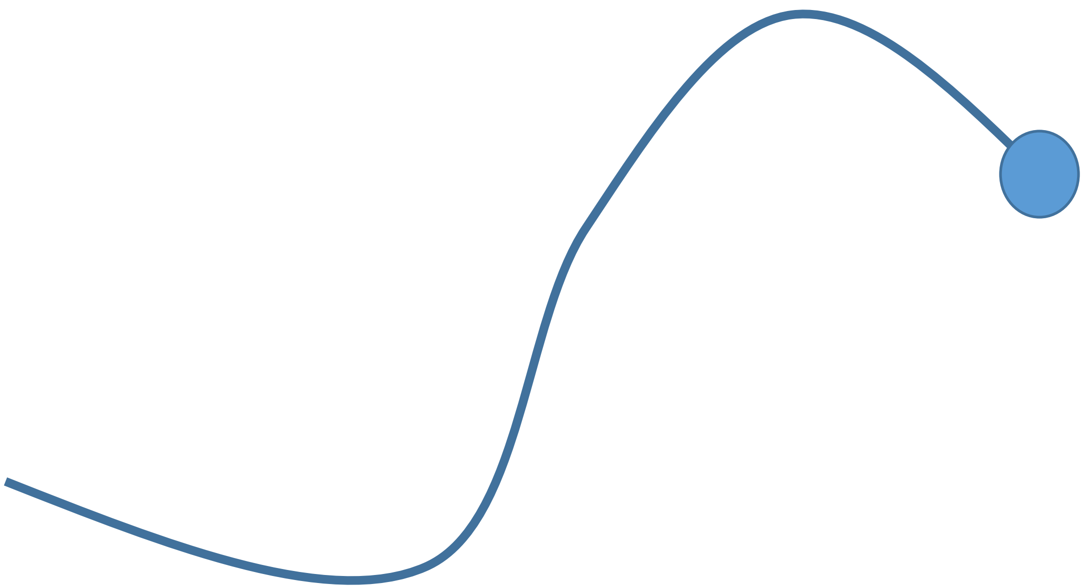
Robot Interaction

- Robot is at a state x_t
- Takes a measurement z_t
- Takes action u_t
- Transitions to a new state x_{t+1}

Actuation

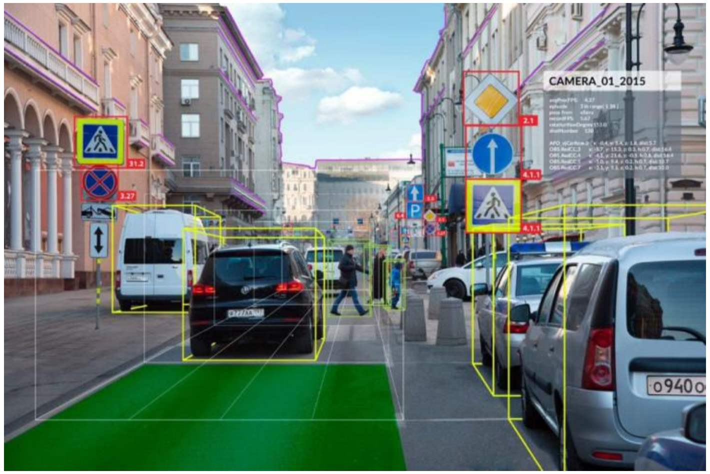
$$p(x_t | x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t | x_{t-1}, u_t)$$

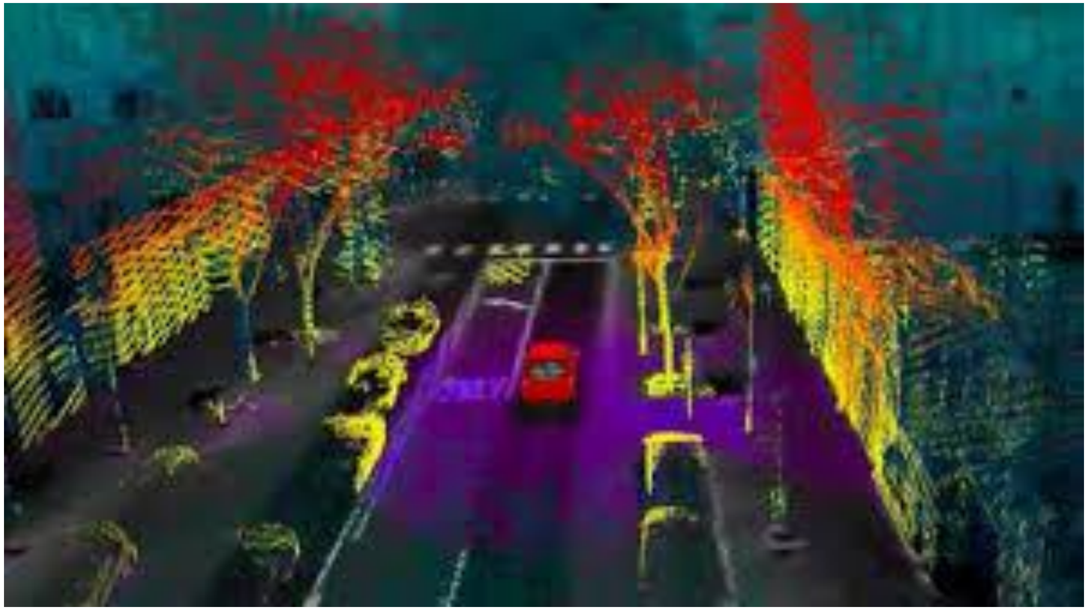
(Markov assumption)



Sensing

- Camera images
- Range scan
- Tactile sensors
- Others?





Sensing

$$p(z_t | x_{1:t}, z_{1:t-1}, u_{1:t}) = p(z_t | x_t)$$

(**Markov assumption**)