Lecture 10: Configuration Spaces & Homogenous Transformations

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1 Configuration Spaces

Robot configuration is the position of every point in that system. The configuration space Q includes all possible configurations of the robot. Figure 1 shows a two-DOF planar manipulation, in which each link is rotating about it's joint (revolute joint). These rotation angles (q_1 and q_2) are the configurations of the robot.

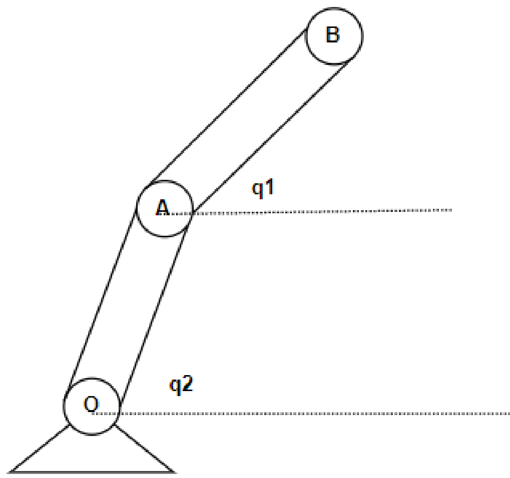


Figure 1: The angles q_1 and q_2 specify the configuration of the two-joint robot.

Each of this joint angles is a point on the unit circle S^1 (Fig. 2). We are identifying this as configuration $q_i \in [0, 2\pi]$. The configuration space in this case is $S^1 \times S^1 = \mathbb{R}^2$.

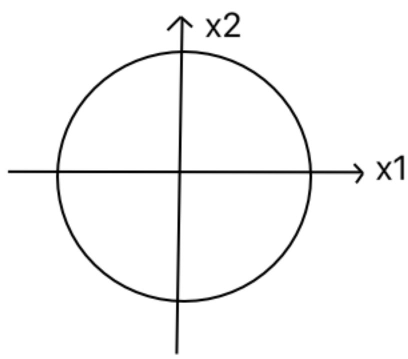
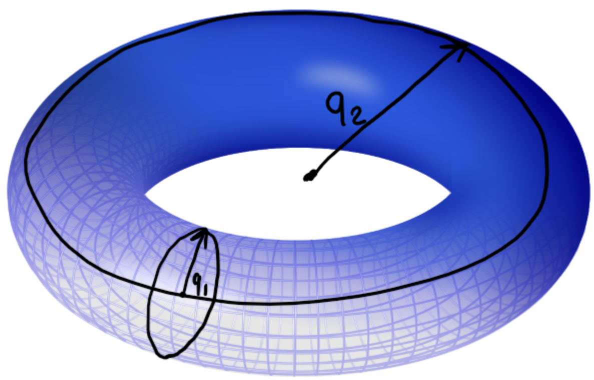
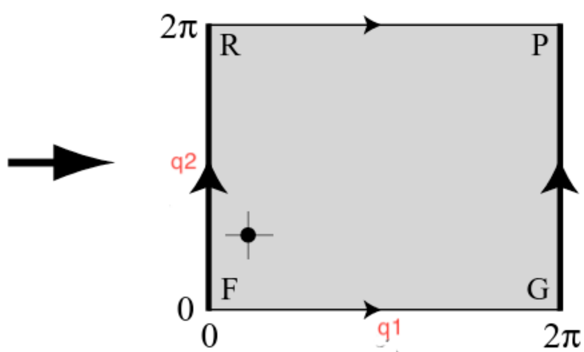


Figure 2: q_i can be represent as a unit circle. $||x|| = 1$

We usually show this as a torus, because it has a natural embedding in $\mathbb{R}^3$. If we cut the torus along the $q_1 = 0$ and $q_2 = 0$ curves, we can flatten the torus onto a plane (Fig. 3).



(a)



(b)

Figure 3: (a) Torus configuration space for 2-DOF planar manipulator, (b) this torus can be cut and flattened onto the plane.

$$R(x_1, x_2) = \{(x_{i1}, x_{i2}) | (x_1 - x_{i1})^2 + (x_2 - x_{i2})^2 \leq r^2\} \tag{1}$$

2 Path Planning Problem

If Q represents a topological space, then a path is $\tau : [0, 1] \rightarrow Q$

2.1 Configuration Space Obstacles

A configuration space obstacle (QO_i) can be defined by the equation:

$$QO_i = \{q \in Q \mid R(q) \bigcap WO_i \neq \emptyset\} \tag{2}$$

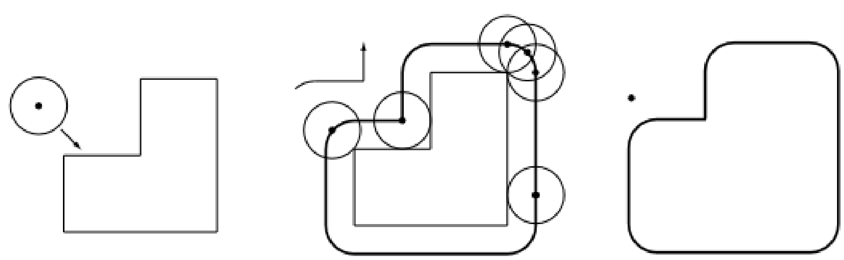
or the set of configurations where the robot intersects an obstacle WO_i
This can provide us with the set of configurations where the robot does not intersect any of the obstacles, or the free space Q_{free}:

$$Q_{free} = Q \setminus (\bigcup_i QO_i) \tag{3}$$

This allows us to then define a *free path* to be a continuous mapping: $\tau : [0,1] \rightarrow Q_{free}$

2.2 Circular Mobile Robot Example

We are able to think of the robot as a single point in the space if we enlarge the obstacles in the space based on the size of the robot. For example, if we imagine an environment where there is a single circular robot as well as one polygonal obstacle, we can increase the size of the obstacle by moving the robot around that obstacle so that it is just touching the edge of the obstacle. An illustration of this is shown below.



(a) (b) (c)

Figure 4: (a)The Robot Environment. (b) Trace the robot around the obstacle and connect the center points. (c) The result leaves a single point with a larger obstacle

3 Degrees of Freedom (DOF) & Holonomic Constraints

3.1 Definition and Relationship

The *degrees of freedom* (DOF) of a rigid body is defined as the number of independent movements it has. In other words, DOF is the number of independent parameters that

define its configuration.
Respectively, Constraints on the position (configuration) of a system of particles are called *holonomic constraints.*
A constraint condition can reduce the DOF of the system if it can be used to express a coordinate in terms of the others.

3.2 Example

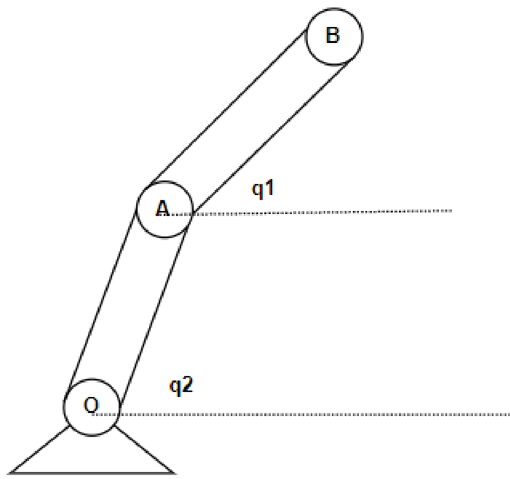


Figure 5: Rods links to a base

Let's assume that

$$O = \begin{bmatrix} 0 & 0 \end{bmatrix}$$
$$A = \begin{bmatrix} x1 & y1 \end{bmatrix}$$
$$B = \begin{bmatrix} x2 & y2 \end{bmatrix}$$

thus

$$R = \begin{bmatrix} x1 & y1 \\ x2 & y2 \end{bmatrix}$$

there are 2 constraints, as length of OA and AB is fixed:

$$(x1 - 0)^2 + (y1 - 0)^2 = l_{OA}^2 \tag{4}$$

$$(x2 - x1)^2 + (y2 - y1)^2 = l_{AB}^2 \tag{5}$$

Recall that system with *n* coordinates and *m* constraints has *n-m* degree of freedom. Thus the whole system's DOF is 4 - 2 = 2.
What if we add another constraints

$$\begin{cases} \frac{y_1}{x_1} = \frac{y_2}{x_2}, x1 \cdot x2 \neq 0 \\ \frac{y1}{y2} = \frac{l_{OA}}{l_{OA} + l_{AB}}, x1 \cdot x2 = 0 \end{cases} \tag{6}$$

This means that angle q1 = q2. Now we have 3 constraints, the DOF is 4 - 3 = 1.

4 Transformation Matrices

4.1 Rotation Matrices

Rotation matrices are used to perform rotations in Euclidean space. They are comprised of an x vector a y vector and a z vector.

$$R = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$

These vectors, such as the x vector below:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

belong to the special orthogonal group SO(3) which have specific properties we can exploit.

$$||x|| = ||y|| = ||z|| = 1 \tag{7}$$

$$x^T = y^T = z^T = 0 \tag{8}$$

Equation (7) utilizes the unit vector aspect. Equation (8) utilizes the orthogonality of the group.

Example

From the planar manipulator case earlier; only one angle is needed hence we use group SO(2).

$$R = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} \cos(q) & -\sin(q) \\ \sin(q) & \cos(q) \end{bmatrix}$$

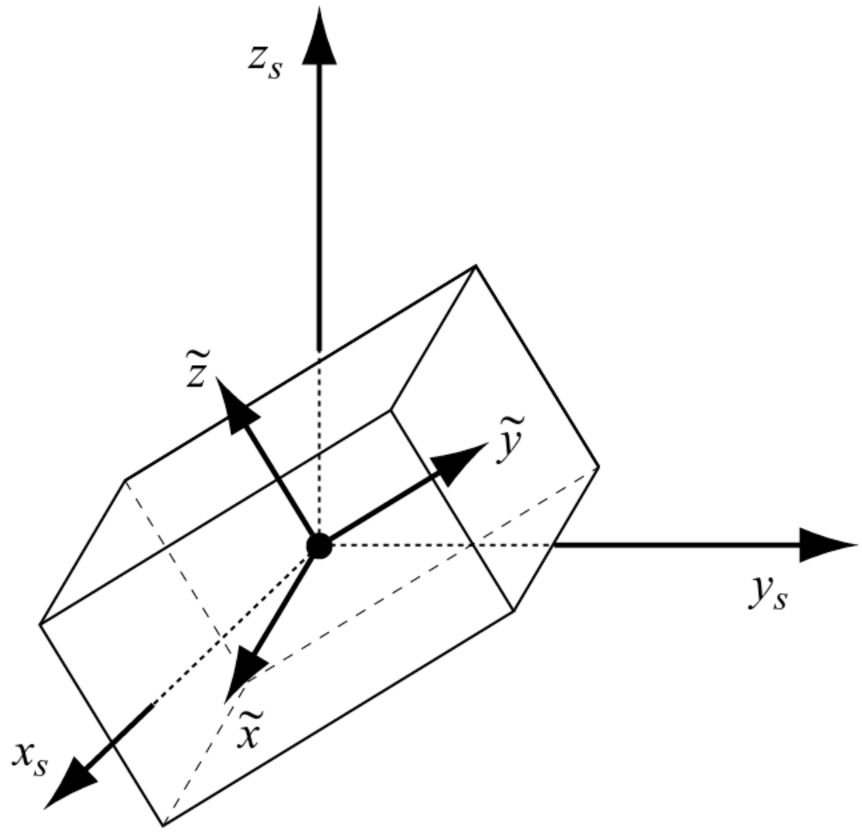


Figure 6: Rotation of a 3-D Rectangle with respect to a coordinate frame.

4.2 Homogeneous Transformations

To represent any position and orientation of an object or a point in 3-D space, we use a general rigid-body Homogeneous Transformation Matrix.
We represent Transformation Matrices in a [4x4] matrix with a convention,

$$T_{AB} = \begin{bmatrix} R_{AB} & P_{AB} \\ 0_{1x3} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & x \\ r_{21} & r_{22} & r_{23} & y \\ r_{31} & r_{32} & r_{33} & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{SE}(3)$$

is the transformation (translation and orientation) from reference frame A to reference frame B comprised of rotation matrix R *a [3 x 3] matrix* and position/translation vector P *a [3 x 1] vector*. Finally we pad the 4th row with a 0 0 0 1 for easier multiplication as demonstrated later on below.

SE is the *Special Euclidean Group*

4.2.1 Representing rigid body configurations & Transforming frames of reference

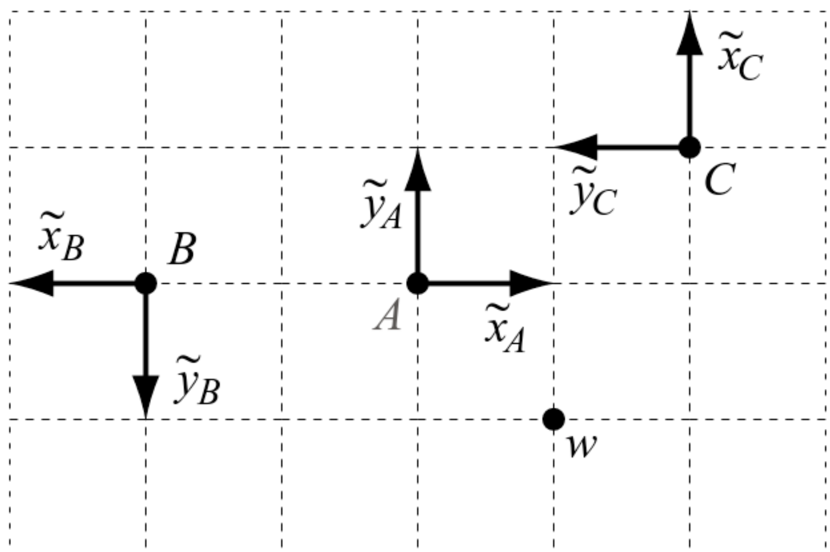


Figure 7: Reference Frame Transformations Example 1

$$T_{AB} = \begin{bmatrix} R_{AB} & P_{AB} \\ 0_{1x3} & 1 \end{bmatrix}; R_{AB} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; P_{AB} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

$$T_{AB} = \begin{bmatrix} R_{AB} & P_{AB} \\ 0_{1x3} & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{BC} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; P_{BC} = \begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix}$$

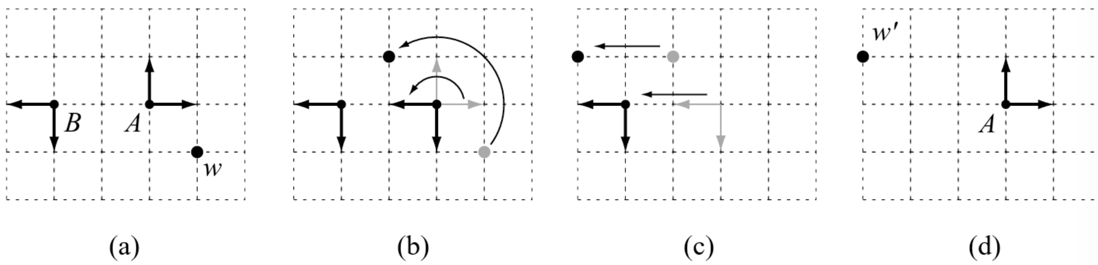


Figure 8: Reference Frame Transformations Example 2.

$$T_{BC} = \begin{bmatrix} R_{BC} & P_{BC} \\ 0_{1x3} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Claim:

$$T_{AC} = T_{AB} * T_{BC} = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.2.2 Displacing a point

[h] First we will begin by transforming a point W from one reference frame to another just as above. Notice that since we are transforming W from reference frame C to B we use the transformation matrix from B to C.

$$W_A = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}; W_C = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}; T_{BC} = \begin{bmatrix} 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$W_B = T_{BC} * W_C = \begin{bmatrix} 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Now lets displace point W relating to its reference frame A to another location still in relation to reference frame A. This is important to notice; we are **not** changing reference frames. We need a transformation matrix to to describe/quantify this displacement. We will use the transformation matrix from reference frame A to B just for convenience, but note **not** as a change of reference frames. *Look closely to the order of lettering versus the example above.*

$$W'_A = T_{AB} * W_A = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$