Lecture 10: *Kinematic Transformations*

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1 Review of Configuration Space

The configuration space, or C-space, of the robot system is the space of all possible configurations of the system. Thus a configuration is simply a point in this abstract configuration space. Some common robots examples and their configuration spaces are given in table 1.

Type of robot	**Representation of Q**
Mobile robot translating in the plane	$\mathbb{R}^2$
Mobile robot translating and rotating in the plane	$\mathbb{R}^2 \times S^1$
Rigid body translating in the three-space	$\mathbb{R}^3$
A spacecraft	$SE(3)$ or $\mathbb{R}^3 \times SO(3)$
An n-joint revolute arm	T^n
A planar mobile robot with an attached n-joint arm	$SE(2) \times T^n$ [9pt]

Table 1: Configuration spaces example

The manifolds S^n, T^n, and $SO(n)$ are different from the $\mathbb{R}^n$ and $SE(n)$ due to whether the manifolds are compact. However, $\mathbb{R}^1$ and $SO(2)$ are one-dimensional manifolds, no matter they are compact or not.

2 Parameterizations of SO(3)

SO(3) can be locally parameterized using three variables. Figure 1 shows Z-Y-Z Euler angels which Euler angles are defined by three successive rotations as follows. As shown in figure 1, The rotation is first performed around the Z axis, with angle ϕ; then around

Y axis, with angle θ; finally around Z axis again, with angle ψ. The rotation around X, Y and Z axis is called pitch, roll, yaw respectively.

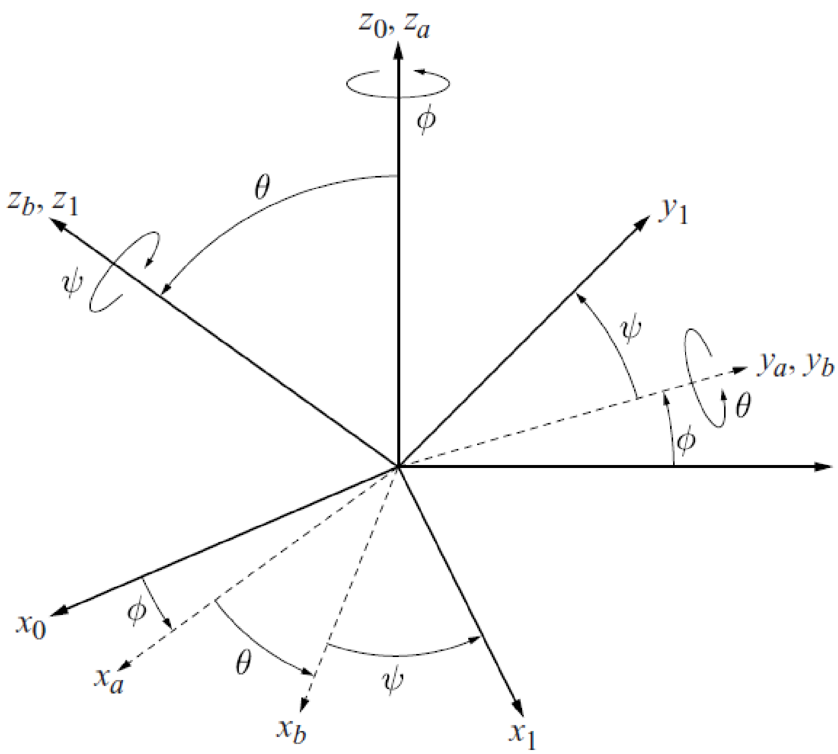

Figure 1: $Z - Y - Z$ rotation transformation.

Rotation matrix is useful for performing rotations. It contains 9 parameters and 6 constraints, which results in 3 degrees of freedom, commonly known as Euler Angles. Some useful equations:

$$w_A = T_{AB} W_B \tag{1}$$

Equation 1 shows how to transfer point w in frame$B$ to in frame A using transformation matrix T_{AB}.

$$w_{A'} = T_{AB} W_A \tag{2}$$

Equation 2 performs coordinate displacement of point w under frame A using transformation matrix T_{AB}. The resulting point $w_{A'}$ will be rotated and translated according to rotation/translation component in T_{AB}.

$$T_{AC} = T_{AB} T_{BC} \tag{3}$$

Equation 3 shows that transformation matrix T_{AC} can be decomposed into a multiplication of a chain of translation matrices T_{AB} and T_{BC}.

3 Transforming Configuration

For n-revolute robot arm, shown as figure 2, the transformation matrix T can be decomposed into the n length multiplication of each joints' transformation matrix.

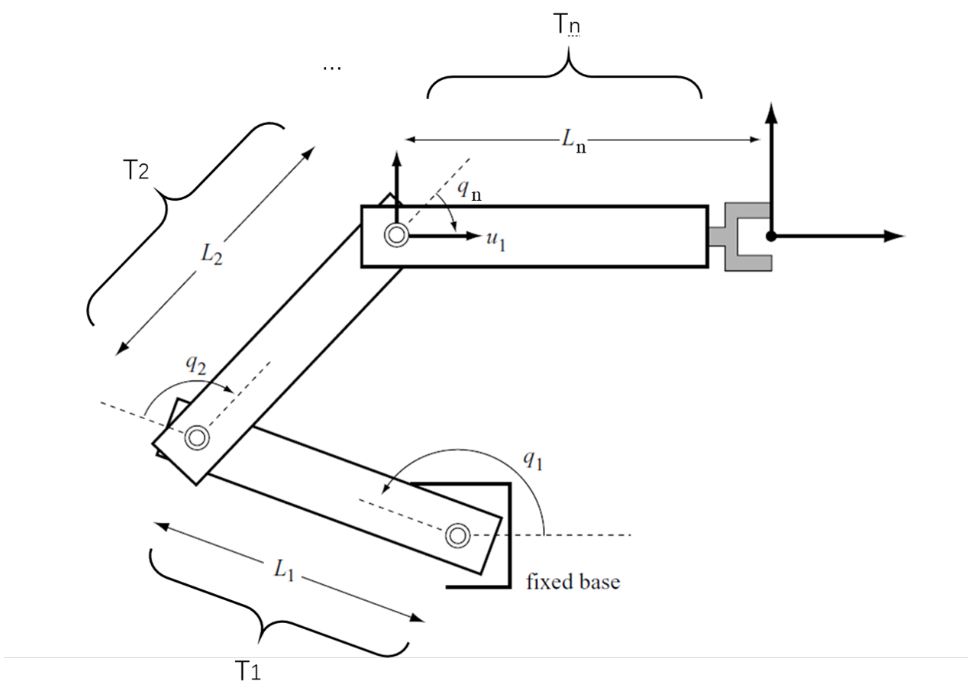

Figure 2: n-R robot arm

T_1 in equation 4 means body frame.

$$T = T_1 ... T_n \tag{4}$$

3.1 Forward Kinematics

Definitions: A mapping from configuration space to a family of Lie Group: $Q \to SE(3)$.

Explanation: By using kinematic equations, we can compute the position of the robot's end-effectors from the joint parameters[1].

We will use a 2R robot arm as an example. Figure 3 shows an 2R robot arm. From the last lecture, we can get the transformation matrices as following.

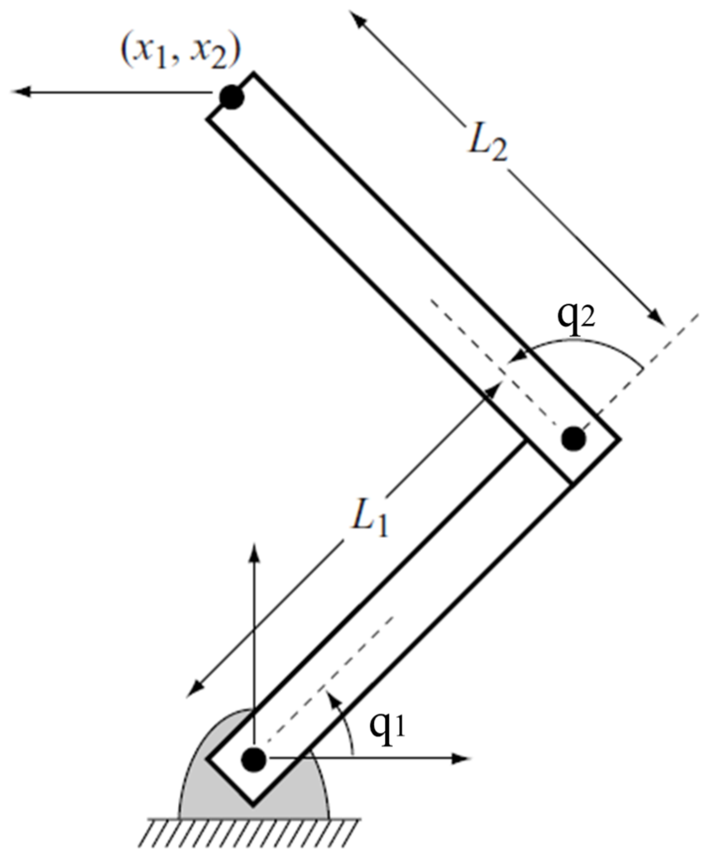

Figure 3: 2-R robot arm

$$T_1 = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 & l_1 \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$T_2 = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 & l_2 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & l_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$T = T_1 \cdot T_2 = \begin{bmatrix} c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 & 0 & l_2c_1c_2 - l_2s_1s_2 + l_1c_1 \\ s_1c_2 + s_2c_1 & -s_1s_2 + c_1c_2 & 0 & l_1s_1c_2 + l_2c_1s_2 + l_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

$$R = \begin{bmatrix} c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 & 0 \\ s_1c_2 + s_2c_1 & -s_1s_2 + c_1c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(q_1+q_2) & -\sin(q_1+q_2) & 0 \\ \sin(q_1+q_2) & \cos(q_1+q_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

The R part of transformation matrix could be trans to the Equation 8. In fact, we can directly get the relationship between end-effector frame and robot-base frame by observing the trigonometry. It corresponds to the two-step transformation chain calculated in Equation 7.

3.2 Inverse Kinematics

Definitions: A mapping from a particular Lie Group to configuration space: $SE(3) \to Q$. It's the inverse of forward kinematics.

Explanation: In robotics, inverse kinematics makes use of the kinematics equations to determine the joint parameters that provide a desired position for each of the robot's end-effectors[1].

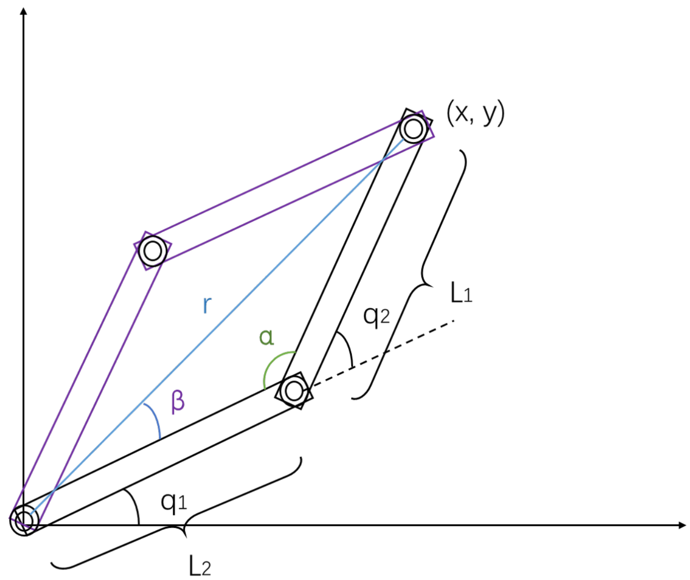

Figure 4: Inverted kinematics for 2R robot arm

$$r = \sqrt{x^2 + y^2} \tag{9a}$$

$$r^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos\alpha \tag{9b}$$

$$\alpha = \pm\cos^{-1}\left(\frac{r^2 - {l_1}^2 - {l_2}^2}{-2l_1 l_2}\right) \tag{9c}$$

$$q_2 = \pi - \alpha \tag{9d}$$

$$q_1 = \text{atan2}\,(x, y) - \beta \tag{9e}$$

$$\beta = \pm\cos^{-1}\left(\frac{r^2 + {l_1}^2 - {l_2}^2}{-2l_1 r}\right) \tag{9f}$$

The equation 9c and equation 9f show different possibilities of α and β. According to figure 4 the inverse kinematics also provide two solutions.

$$\cos\alpha = \frac{r^2 - {l_1}^2 - {l_2}^2}{-2l_1l_2} = \frac{r^2 - (l_1 + l_2)^2 + 2l_1l_2}{-2l_1l_2} = \frac{r^2 - (l_1 + l_2)^2}{-2l_1l_2} - 1 \tag{10}$$

The Equation. 10 disproves the situation $r > l_1 + l_2$ can be feasible.
If $r > l_1 + l_2$ then $\frac{r^2-(l_1+l_2)^2}{-2l_1l_2}$ will less than 0, then $\cos\alpha$ will less than -1, which is impossible.

3.3 Numerical Optimization for Inverse Kinematics

For more complex inverse kinematic problems, it's often impractical to calculate close-form solution based on geometrics, therefore the IK problem is formulated as a numerical optimization problem with the following objective:

$$min \quad J = ||P_{curr} - P_d|| + \lambda||q|| \tag{11}$$

subject to.

$$q_{min} < q_i < q_{max} \tag{12}$$

$$R(q)\bigcap WO_i = \phi \tag{13}$$

$$Q(q) = P_{curr} \tag{14}$$

Equation. 12 means that each joint angle in the configuration space needs to satisfy the joint limits, and that the geometric space of joint configuration should not collide with any obstacles (Equation. 13). Finally and most importantly is that the joint configuration should satisfy the kinematics constraints (Equation. 14). The goal is to minimize the difference between current pose to the target pose, as well as the change in joint configuration (Equation. 11).

References

[1] Richard P Paul. *Robot manipulators: mathematics, programming, and control: the computer control of robot manipulators*. Richard Paul, 1981.