Lecture: *Dynamics*

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1 Introduction: Manipulator Dynamics

In this lecture, we will discuss the dynamics of manipulation and study the forces required to cause the corresponding motion for robotic system. More specifically, we want to find out what torque we should input to the system, so we could achieve a given position or velocity. To compute the torque, we need to take the mass and geometry of the robot into account.

We could potentially use Newton's equations, but they are specified for point mass. Therefore, we want to generalize to rigid body, we will need more sophisticated system.To describe this system, let's say we have a system configuration specified by r coordinates, and forces:

$$Configurations : q_1, q_2...q_r$$

$$Generalized\ forces : Q_1, Q_2...Q_r$$

By using the generalized coordinates and forces, we can define work as the sum of generalized force applied to each the generalized coordinates times the small displacement in that coordinate.

$$W = \sum_{i=1}^{r} Q_i dq_i$$

Let's look at an example. Let's say if we have N=5 particles in 3-dimensional space and ignore any constraints, we will have 3N coordinates to represent the system. If we have k constraints:

$$f_j(r_1, r_2...r_n) = 0 \quad j = 1, ..., k$$

$$x_i = x_j + c$$

In this case, we will need 3N-k coordinates to define the system. Therefore, We call the q_i's a set of generalized coordinates for the system. The specification determines the position of all of the particles which make up the robot. For a system of n particles with k constraints, we seek a set of m = 3n-k variables $q_1, ..., q_m$ and functions $f_1, ...f_n$ such that:

$$r_i = f_i(q_1, ..., 1_m)$$

$$m = 3n - k$$

$$i = 1, ..., n$$

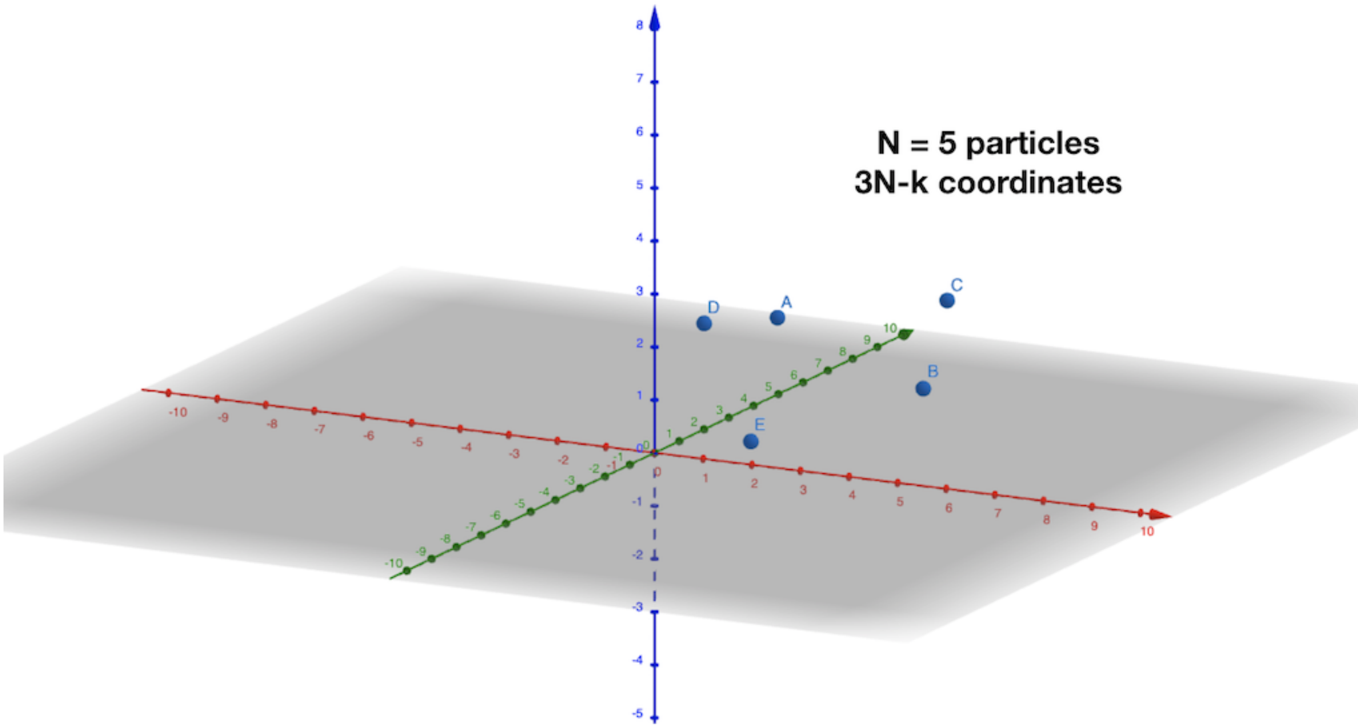

Figure 1: An example with 5 particles in a 3-dimensional space

Now let say there is force acting on these particles. we can define *total work* as the following:

$$W = \sum_{i=1}^{N} f_i.dr_i$$

This is sum of forces applied to each particle times displacement of that particle. Next, we need to express this in terms of general coordinates. We use *chain rule* to expand as following:

$$W = \sum_{i=1}^{N} f_i \sum_{j=1}^{3N-k} \frac{dr_i}{dq_j} dq_j$$

By changing the order of the summations we have:

$$W = \sum_{j=1}^{3N-k} \underbrace{\sum_{i=1}^{N} f_i \frac{dr_i}{dq_j} dq_j}_{Q_j}$$

where:

- Q_j: Generalized force is not in Cartesian Coordinate system (General Coordinate system)

2 Lagrangian Formulation of Manipulator Dynamics

If we start with Newtonian Mechanics and apply this transformation, we can derive a generalized form of equations of motion called Lagrangian Mechanics which is defined in generalized coordinate system and in generalized forces terms. We are going to skip the details here. We can define Lagrangian $\mathcal{L}$ as the difference between *Kinetic Energy* and *Potential Energy*:

$$\mathcal{L} = \underbrace{T}_{KineticEnergy} - \underbrace{U}_{PotentialEnergy}$$

Then we can say that rate of change (partial derivative) of $\mathcal{L}$ with respect to generalized velocities minus rate of change of $\mathcal{L}$ with respect to generalized positions equals the amount of generalized external forces applied to the system. In other words if we represent the total kinetic energy of a system with respect to some generalized coordinate system, the total change in that energy over time equals the sum of external forces applied to that system.

We can formulate the statement above as following:

$$\frac{d}{d_t}(\frac{dL}{d\dot{q}_i}) - \frac{dL}{dq_i} = Q_i$$

Example: We can define kinetic energy of one particle as:

$$T = \frac{1}{2} m v^2$$

if we re-write this in q term we have:

$$T = \frac{1}{2} m \dot{q}^2$$

where $\dot{q} \equiv \dot{x}$ represents velocity therefore $q \equiv x$ represents position.

If we assume potential energy $U = 0$ in Lagrangian equation and replace T with the above equation, we have:

$$\frac{dL}{d\dot{q}} = m\dot{x}$$
$$\frac{d}{dt}(\frac{dL}{d\dot{q}_i}) = m\ddot{x}$$
$$\frac{dL}{dq_i} = 0$$

So we showed $\mathcal{L}$ = mass $\times$ acceleration which is equivalent to Newton's second law. Our goal is to use Lagrangian equations to derive equations of motion for a manipulator. We need to establish a relationship between angles, angular velocities and the torques. The reason is that in real world the control command to the robot joints (motor) will be in terms of **T** (Torque).

3 Derivation of Kinetic Energy for Manipulator links

First, let us introduce some definitions and model the robot link mass distribution.

- $V \in \mathbb{R}^3$: Volume
- $\rho(r)$: Mass Distribution
- m: $\int_V \rho(p)dV$

Therefore we can define the vector $\vec{P_c}$ representing center of mass as following:

$$\vec{P_c} = \frac{1}{m}\int_V \rho(p).pdV$$

Figure.2 shows object's mass distribution. Below are some definitions related to this figure:

- $\vec{\omega}$: Angular Velocity around center of Mass
- $\vec{r} = \vec{p_c} - \vec{p}$

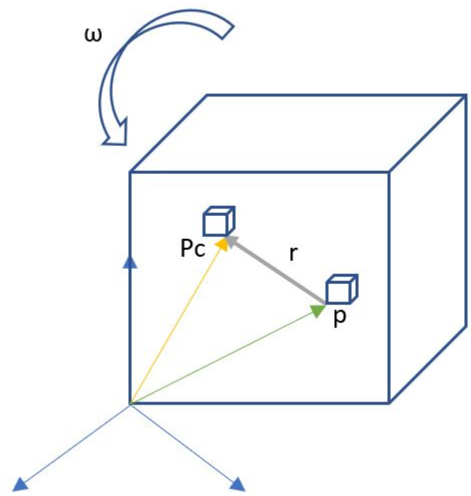

Figure 2: Mass distribution

- $\dot{p} = \dot{p}_c + \vec{\omega} \times \vec{r}$
  where $\dot{p}_c$: Linear Velocity of Center of Mass

Since ω has three components (one for each axis), this equation will be re-written as following:

$$\dot{p} = \dot{p}_c + S(\omega) \times \vec{r}$$

where

$$S(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

We know that kinetic energy is equal to 1/2 the product of the mass and the square of the speed. In formula form :

$$T = \frac{1}{2} m v^2$$

Thus, kinetic energy of one particle :

$$T = \frac{1}{2} \rho \dot{p}^T \dot{p}$$

where ρ is mass distribution and we assume it's constant and $\dot{p}$ is the velocity. The reason to have $\dot{p}$ transpose is just for matrix multiplication.

Then kinetic energy of a link is to integral up all the particles on the link. Thus the formula form :

$$T = \int_V \frac{1}{2} \rho \, \dot{p}^T \dot{p} \, dV$$

Then we replace $\dot{p} = \dot{p}_c + S(\vec{\omega})\vec{r}$, where $\dot{p}_c$ is linear velocity of center of mass, $\vec{\omega}$ is rotational speed, and $\vec{r}$ is the vector from center to particle p. Then, we expand it.

$$T = \int_V \frac{1}{2} \rho \{ \parallel \dot{p}_c \parallel^2 + \underbrace{2\, \dot{p}_c \, S(\vec{\omega})\, \vec{r}}_{\text{second term}} + \underbrace{[S(\vec{\omega})\, \vec{r}]^T [S(\vec{\omega})\, \vec{r}]}_{\text{last term}} \} dV \tag{1}$$

Second term of formula 1 :

$$\begin{aligned} \int_V \frac{1}{2} \rho \{ 2\, \dot{p}_c \, S(\vec{\omega})\, \vec{r} \} dV &= \dot{p}_c \, S(\vec{\omega}) \int_V \rho \, \vec{r} \, dV \\ &= \dot{p}_c \, S(\vec{\omega}) \underbrace{\int_V \rho \, (p - p_c) \, dV}_{\text{equal to } 0} \\ &= 0 \end{aligned} \tag{2}$$

Now we explain why $\rho \int_V (p - p_c)\, dV = 0$, recall the formula of p_c:

$$\begin{aligned} p_c &= \frac{1}{m} \int_V \rho \, p \, dV \\ &= \frac{\int_V \rho \, p \, dV}{\int_V \rho \, dV} \end{aligned}$$

From the above formula we can infer that :

$$\begin{aligned} \int_V \rho \, p_c \, dV &= \int_V \rho \, p \, dV \\ \Rightarrow \rho \int_V (p - p_c) \, dV &= 0 \end{aligned}$$

Thus, second term is equal to 0.

Before start explain the last term of formula 1, we recall that cross-product is anticommutative :

$$\begin{aligned} \vec{\omega} \times \vec{r} &= -\vec{r} \times \vec{\omega} \\ \Rightarrow S(\vec{\omega}) \times \vec{r} &= -S(\vec{r}) \times \vec{\omega} \end{aligned}$$

Last term of formula 1 :

$$\begin{aligned} \frac{1}{2}\int_V \rho[S(\vec{\omega})\,\vec{r}]^T[S(\vec{\omega})\,\vec{r}]\,\}\,dV &= \frac{1}{2}\int_V \rho[-S(\vec{r})\,\vec{\omega}]^T[-S(\vec{r})\,\vec{\omega}]\,dV \\ &= \frac{1}{2}\int_V \rho\,\vec{\omega}^T S(\vec{r})^T S(\vec{r})\,\vec{\omega}\,dV \\ &= \frac{1}{2}\,\vec{\omega}^T[\int_V \rho\,S(\vec{r})^T S(\vec{r})\,dV]\,\vec{\omega} \\ &= \frac{1}{2}\,\vec{\omega}^T I\,\vec{\omega} \end{aligned} \tag{3}$$

Where I is inertia matrix.

Combine formula 1, 2 and 3, we can show that total kinetic energy of a link :

$$T = \frac{1}{2}\,m \parallel \dot{p}_c \parallel^2 + \frac{1}{2}\,\vec{\omega}^T I\,\vec{\omega}$$