Lecture #23: *Dynamics of 2 DOF Manipulator*

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1 Recap: Dynamics of Robotic Manipulators

In the previous lecture, we discussed derivation of kinetic energy for manipulator link, kinematic relation for link is:

$\dot{p} = \dot{p_c} + S(\omega)r$, $\dot{p} = \dot{p_c} + r$....(1)

where, $S(\omega)$ is skew-symmetric matrix and ω is angular velocity, p is velocity, p_c is the velocity of the center of mass of the rigid body.

And we discussed kinetic energy(T) of link:
$T = \int_v \frac{1}{2}\rho\dot{p}^T\dot{p}dv$..... (2)
$\rho(r)$ is mass distribution ; mass $m = \int_v \rho(p)dv$

2 König Theorem

In kinetics, König's theorem or König's decomposition is a mathematical relation derived by Johann Samuel König that assists with the calculation of kinetic energy of bodies and systems of particles. The theorem can be applied to rigid bodies(manipulator link), stating that the kinetic energy T of a rigid body(manipulator link) can be written as:

$T_i = \frac{1}{2}m_i v_i^2 + \frac{1}{2}\omega_i^T I_i \omega_i$

where m is the mass of the rigid body; v is the velocity of the center of mass of the rigid body and is the angular velocity of the rigid body. $\omega_i I_i$ should be expressed in the same reference frame, but the product $\omega_i^T I_i \omega_i$ is invariant w.r.t. any chosen frame.

The above formula can be obtained by the formula (1) and (2) from the recap section.

$T = \int_v \frac{1}{2}\rho\dot{p}^T\dot{p}dv$ where p is the velocity and v is the mass of the rigid body.

$= \frac{1}{2}\int_v \rho(||\dot{p}||^2 + 2\dot{p_c}S(\omega)r + (S(\omega)r)^T(S(\omega)r))dv$ where p_c is the velocity of the centre of the mass of the rigid body

- $\frac{1}{2}\int_v \rho||\dot{p}||^2 dv = \frac{1}{2}p_c^2 v$ which is the translational kinetic energy (point mass of CoM)
- $\int_V \rho(\dot{p_c}S(\omega)r)dv = \dot{p_c}S(\omega)\int_v r dv = 0$
- $\frac{1}{2}\int_v \rho((S(\omega)r^T(S(\omega)r)))dv = \frac{1}{2}\int_v \rho((-S(r)\omega)^T(-S(r)\omega))dv$ since $\omega x r = r x \omega$ and $S(\omega)r = -S(r)\omega$
  = $\frac{1}{2}\int_v \rho\omega^T S(r)^T S(r)\omega dv$
  = $\omega^T \int_v \rho S(r)^T(r)dv\omega \int_v \rho S(r)^T(r)dv$
  = $\frac{1}{2}\omega^T I\omega$ rotational kinetic energy (of the whole body)

Then we obtained the formula $T_i = \frac{1}{2}m_i v_i^2 + \frac{1}{2}\omega_i^T I_i \omega_i$

3 Dynamics for 2 DOF Manipulator

There is a two-link robot arm shown as below:

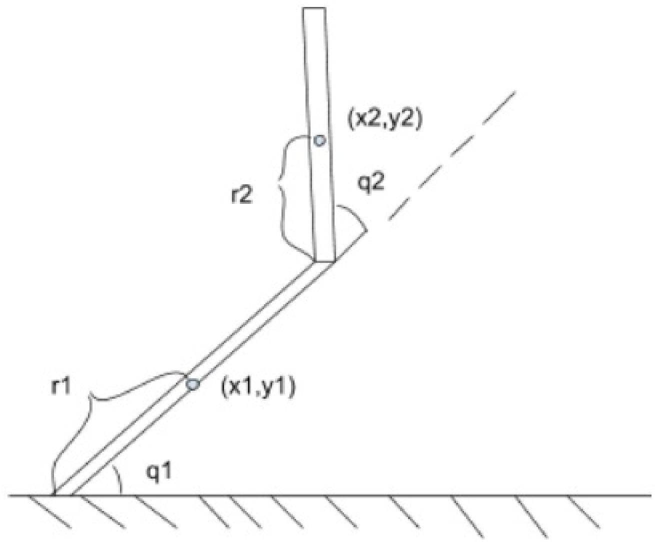


Figure 1: Two link Robot Arm

According to König's theorem:

$T_i = \frac{1}{2}m_i v_i^2 + \frac{1}{2}\omega_i^T I_i \omega_i$

where m_i is the mass of the i^{th} link; v_i is the velocity of the center of mass of the i^{th} link and ω_i is the angular velocity of the i^{th} link.

So total kinetic energy T for the above two-link robot arm:

$T_i = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}\omega_1^T I_1 \omega_1 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}\omega_2^T I_2 \omega_2$

$= \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}\dot{p_1}^T I_1 \dot{p_1} + \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2) + \frac{1}{2}[0, 0, \dot{p_1} + \dot{p_2}] I_2 [0, 0, \dot{p_1} + \dot{p_2}^T]$

where (x_i, y_i) represents the position of the center of mass of the i^{th} link; q_i is the angular between i^{th} link and $(i-1)^{th}$ link(q_1 is the angular between 1^{st} link and ground).

Based on structure of robot arm, there exist equations below:

$x_1 = r_1 c_1$

$x_2 = l_1 c_1 + r_2 c_{12}$

$y_1 = r_1 s_1$

$y_2 = l_1 s_1 + r_2 s_{12}$

$\dot{x_1} = -r_1 s_1 \dot{q_1}$

$\dot{x_2} = -l_1 s_1 \dot{q_1} - r_2 s_{12}(\dot{q1} + \dot{q2})$
$\dot{y_1} = r_1 c_1 \dot{q_1}$
$\dot{y2} = l_1 c_1 \dot{q_1} + r_2 c_{12}(\dot{q_1} + \dot{q_2})$

where c_1 is cosine value of q_1; s_1 is sine value of q_1; c_{12} is the cosine value of $q_1 + q_2$; s_{12} is the sine value of $q1 + q2$.

According to Lagrangian:

$L = T - V$

where T is kinetic energy; V is potential energy.

Euler-Lagrange equations for two-link system:

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_i}} - \frac{\partial L}{\partial q_i} = \tau_i$ for $i = 1, 2$

where i is Torque – generalized force performing work on q_i.

Through all equations above, we can get:

$B(q_1, q_2)[\ddot{q_1}, \ddot{q_2}]^T + C(q_1, q_2, \dot{q_1}, \dot{q_2})[\dot{q_1}, \dot{q_2}] + G(q_1, q_2) = [\tau_1, \tau_2]^T$

For general case:

$B(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau$ where B is inertia matrix; C is centrifugal and coriolis forces; G is gravitational forces.

4 Direct and Inverse Dynamics

4.1 Direct Dynamics

The forward (direct) dynamics problem consists of determining for $t > t_0$, the acceleration $\ddot{q}(t)$, velocity $\dot{q}(t)$ and position $q(t)$ resulting from torque $\tau(t)$ and possible end-effector forces $h_e(t)$ once the initial position and velocity of the system at time t_0 are known. Solving the forward dynamics equations is useful for manipulator simulation.

Forward dynamics allows the motion of the real physical system to be described in terms of joint accelerations when a set of assigned joint torques is applied to the manipulator; joint velocities and positions can be obtained by integrating the system of non-linear differential equations.

For simulation of manipulator motion, once the state at the time instant t_k is known in terms of acceleration $\ddot{q}(t_{k+1})$, then using Euler Integration method with integration step of Δt, the velocity $\dot{q}(t_{k+1})$ and position $q(t_{k+1})$ at time instant $t_{k+1} = t_k + \Delta t$ can be computed as follows:

$\dot{q}(t_{k+1})$ = $\ddot{q}(t_k)\Delta t$ + $\dot{q}(t_k)$ .... (a)
$q(t_{k+1})$ = $\dot{q}(t_k)\Delta t$ + $q(t_k)$ .... (b)

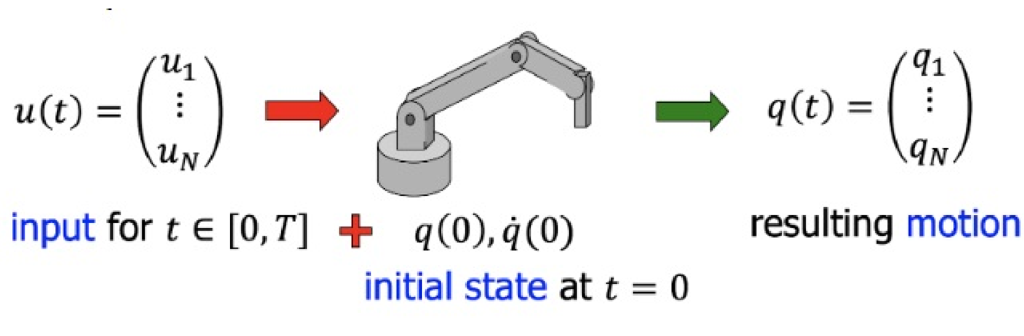


Figure 2: Direct Dynamics

Runge-Kutta method is used to numerically integrate ordinary differential equations by using a trial step at the midpoint of an interval to cancel out lower-order error terms. This method is used quite widely because it is reasonably simple and robust.

If the system's torque τ, inertia matrix B, gravitational forces G and centrifugal and Coriolis forces C is known, then we can calculate acceleration $\ddot{q}(t_d)$ (desired acceleration) as:

$\ddot{q}(t_{k+1}) = B^{-1}[(\tau - C - G)]$........ (c)

However, this may cause problem if the links of the manipulator are too small or too large.

4.2 Inverse Dynamics

The inverse dynamics problem consists of determining the torque $\tau(t)$ which is required to generate the motion specified by acceleration $\ddot{q}(t)$, velocity $\dot{q}(t)$ and position $q(t)$ once the possible end-effector forces $h_e(t)$ are known. Solving inverse dynamics is useful for manipulator trajectory planning and control algorithm implementation.

Once a joint trajectory is specified in terms of positions, velocity and acceleration (typically as results of inverse kinematics procedure) and if end-effector forces are known, inverse dynamics allows computation of torques to be applied to joints to obtain desired motion. This computation is useful both for verifying feasibility of imposed trajectory and also for compensating non-linear terms in dynamic model of manipulator.

For a n-joint manipulator, the number of operations required for computing inverse dynamics is: $O(n)$

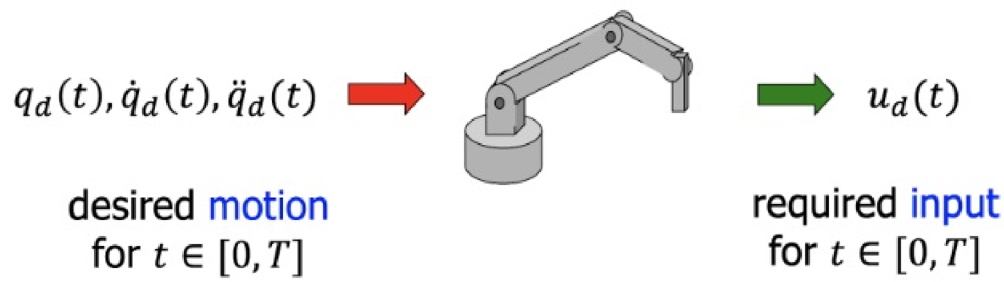


Figure 3: Inverse Dynamics

5 Control

5.1 Decentralized Approach

Decentralized Approch is basically used to calculate $\tau(InverseDynamics)$.

In decentralized approach, we consider few things such as

- If we have a common term, we treat them as disturbances.
- We assume that the link moves independently
- Has feedback and feedforward loop

The equation is given as :

$\tau = K_P(q_d - q) - K_D(\dot{q_d} - \dot{q})$.....(d)

In this approach, we apply a PD control separately for each joint. A better approach however is the Centralized Approach which takes into account the dynamics of the system.

5.2 Centralized Approach

The centralize approach takes care of the dynamics. The equation is given as :

$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q_d)\ddot{q_d} + C(q_d,\dot{q_d})\dot{q_d} + G(q_d) + K_P(q_d - q) + K_D(\dot{q_d} - \dot{q})$

To take into account error between desired and current, consider $\tilde{q}$= $q_d - q$ Then,

B(q)$\ddot{\tilde{q}} + C(q,\dot{q})\dot{\tilde{q}} = -K_p\tilde{q} - K_d\dot{\tilde{q}}$

If q_d, q are very close then $B(q) \approx B(q_d)$.

$B(q)\ddot{\tilde{q}} + [C(q,\dot{q}) + K_D]\dot{\tilde{q}} + K_p\tilde{q} = 0$

5.3 Example: Gravity Compensation

The gravity terms will tend to cause static positioning errors, so some robot manufacturers include a gravity model, G(q), in the control law. The complete control law takes the following form:
$\tau = G(q) + K_p(q_d - q) - K_D\dot{q}$

Lyapunov's method is used for examining the stability of a system. Lyapunov function is a function of energy of the system. Lyapunov's method is concerned with determining the stability of a system. To prove a system stable by Lyapunov's method, one is required to propose a generalized energy function $V(x)$ that has the following properties:

- $V(x) = 0$ if and only if $x = 0$.
- Energy of the system should be positive i.e. $V(x) > 0$ for all $x \neq 0$
- Derivative of energy of the system should be negative $\dot{V}(x) \leq 0$. Here, $\dot{V}(x)$ means change in $V(x)$ along all system trajectories.

For a function of two variables x_1, x_2 , example of $V(x)$ is schematically shown below:
Consider candidate Lyapunov Function:
$V(\dot{q}, q) = \frac{1}{2}\dot{q}^T B\dot{q} + \frac{1}{2}\tilde{q}V_p\tilde{q}$

$\frac{dV}{dt} = \frac{1}{2}\dot{q}^T B\dot{q} + \frac{1}{1}\dot{q}\dot{B}\dot{q} + \frac{1}{2}\dot{q}^T B\ddot{q} + \tilde{q}^T K_D(-\dot{q})$

To find acceleration: $B\ddot{q} = \tau - G - C$

$\frac{dV}{dt} = \dot{q}^T(\tau - G - C) + \frac{1}{2}\dot{q}^T B\dot{q} - \dot{q}K_p\tilde{q}^T$

If A is symmetric matrix, $x = x^T Ax$

$\frac{\partial \alpha}{\partial x} = \partial x^T A$

$\dot{B} - 2C = 0$ (Holds true for any rigid body mechanics by preservation of energy)

Solving the equations further

$\frac{dV}{dt} = \dot{q}\tau - \dot{q}G - \dot{q}K_p\tilde{q}$

For stability to make the above equation negative, only torque can be controlled thus show that

$\dot{q}\tau - \dot{q}G - \dot{q}K_p\tilde{q} < 0$

$\tau - G - K_p\tilde{q} < 0$

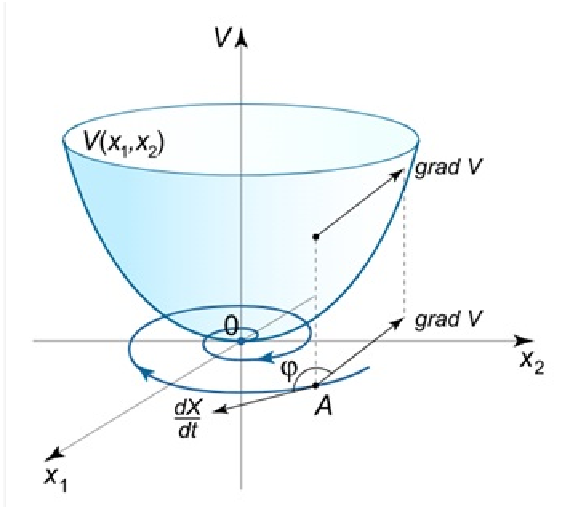
h! 

Figure 4:

References

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